

Random Matrices and the Anderson Model

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Abstract

In recent years, constructive field techniques and the method of renormalization group around extended singularities have been applied to the weak coupling regime of the Anderson Model. It has allowed to clarify the relationship between this model and the theory of random matrices. We review this situation and the current program to analyze in detail the density of states and Green's functions of this model using the supersymmetric formalism.

1 Introduction

This small review is devoted to the elementary theory of random matrices and to the link between this theory and the Anderson model of localization/diffusion of a quantum particle in a random potential.

More precisely we recall first the basic result of random matrix theory, namely the Wigner's semi-circle law for density of states, and give its rigorous supersymmetric derivation.

Then we review the Anderson model, introducing the phase space approach to this model pioneered by Gilles Poirrot, and summarizing the results of [1].

Finally in the last part we propose some generalizations of the flip random matrix model of [1] which are closer to the real Anderson model, using in particular some hierarchical approximations. The control of such more realistic random matrices models is an important step towards rigorous theorems about the Anderson model in the weak potential phase.

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2 Random Matrices and Wigner's law

2.1 The GUE

The simplest ensemble of random matrices is the Gaussian unitary ensemble. It is a probability measure on random hermitian $N \times N$ complex matrices. Each coefficient in the upper triangle of the matrix is identically independently Gaussian distributed. Here the matrix is $H = H_{ij}$, $H = H^*$, hence $H_{ij} = \bar{H}_{ji}$, and

$$P(H) = \frac{1}{Z} \exp\left(-\frac{N}{2} \text{Tr} H^* H\right) \quad (2.1)$$

Z being a normalization factor. The matrix H is made therefore of $N(N-1)/2$ complex variables H_{ij} with $i < j$ and N real ones H_{ii} , so there are N^2 real random variables in H .

Since

$$\text{Tr} H^* H = \sum_{i=1}^N \sum_{j=1}^N |H_{ij}|^2 = \sum_i H_{ii}^2 + 2 \sum_{i < j} (\Re H_{ij}^2 + \Im H_{ij}^2) \quad (2.2)$$

we have

$$Z = 2^{N/2} (\pi/N)^{N^2/2} \quad (2.3)$$

and the covariance rule is

$$\langle H_{ij} H_{i'j'}^* \rangle = \frac{1}{N} \delta_{ii'} \delta_{jj'}. \quad (2.4)$$

The scaling factor $\frac{1}{N}$ has been chosen to keep the typical eigenvalues of H of size $O(1)$ as $N \rightarrow \infty$; indeed the typical size of the eigenvalues of a random matrix with covariance 1 is obviously of order \sqrt{N} , by the law of large numbers.

Physicists would like to know the statistics of the eigenvalues of H , and they are particularly interested in the two first moments of their distribution, called the density of states and the two-level correlation function.

The density of states, $\nu(E)$ for a Hermitian matrix H is the quantity which, when integrated from $-\infty$ to A , counts the number of eigenvalues of H which are lower or equal to A . Since H has exactly N real eigenvalues $\lambda_1, \dots, \lambda_N$, we have

$$\nu(E) = \frac{1}{N} \text{Tr} \delta(E - H) \quad (2.5)$$

so that $\int_{-\infty}^{+\infty} \nu(E) dE = 1$. We can use the standard formula for the Dirac distribution

$$\delta(x - a) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0_+} \Im \frac{1}{x - a + i\varepsilon}. \quad (2.6)$$

Hence

$$\nu(E) = -\frac{1}{\pi N} \lim_{\varepsilon \rightarrow 0_+} \Im \text{Tr} \frac{1}{E - H + i\varepsilon}. \quad (2.7)$$

Physicists call $(E - H \pm i\varepsilon)^{-1}$ respectively the retarded and advanced Green's functions for the Hamiltonian H .

The averaged density of states $\langle \nu(E) \rangle$ is therefore

$$\langle \nu(E) \rangle = - \lim_{\varepsilon \rightarrow 0_+} \int P(H) dH \frac{1}{\pi N} \Im \text{Tr} \frac{1}{E - H + i\varepsilon}, \quad (2.8)$$

and $\langle \nu(E) \rangle dE$ clearly represents the probability for an eigenvalue of H to lie between E and $E + dE$, with normalization condition $\int \langle \nu(E) \rangle dE = 1$.

The main results on the GUE ensemble is Wigner's semi-circle law:

$$\lim_{N \rightarrow \infty} \langle \nu(E) \rangle = \frac{\chi_{|E| \leq 2}}{\pi} \sqrt{1 - E^2/4}. \quad (2.9)$$

The corresponding curve is really a semi-ellipse, but of course could be changed into a circle through a slight reparametrization of the covariance of H . The normalization taken here corresponds to $\int E^2 \nu(E) dE = 1$.

Wigner's law is a central result. It has been called the non-commutative analog of the Gaussian law of large numbers [2], and has been proved to hold in much more general cases than the GUE, for instance for band random matrices [3].

The next quantity of interest is the 2-level correlation, which allows to know the conditional probability to find an eigenvalue of H near E knowing already that one eigenvalue sits at E . More precisely it gives the probability to have two eigenvalues separated by an interval of width ω centered at E , and is therefore

$$R_2(\omega) = \frac{\langle \nu(E - \omega/2) \nu(E + \omega/2) \rangle}{\langle \nu(E) \rangle^2}. \quad (2.10)$$

In the GUE, eigenvalues are not independent but tend to "repel" each other. This is seen in the following behavior of the 2-level correlation R_2

$$R_2(s) = \delta(s) + 1 - \frac{\sin^2 \pi s}{(\pi s)^2}, \quad (2.11)$$

where $s = \omega/\Delta$, and Δ is the mean level spacing $\Delta = 1/N \langle \nu(E) \rangle$. The delta function simply expresses the constraint of presence of an eigenvalue at E . Independence of the eigenvalues would mean $\lim_{s \rightarrow 0} R_2(s) - \delta(s) = 1$, hence no change in the probability for a second value to sit near E if a first is present. But here we have $\lim_{s \rightarrow 0} R_2(s) - \delta(s) = 0$ because of the $\frac{\sin^2 \pi s}{(\pi s)^2}$ term. Hence there is 0 chance for a second eigenvalue to sit near E if a first one sits at E . This is the phenomenon of "eigenvalue repulsion".

Physicists got intuition of this repulsion by the simple observation of the Vandermonde determinant that appears in the Jacobian of the transformation from the initial coefficients of the matrix to the diagonal eigenvalues and the

unitary diagonalizing matrix. In rough terms, we can diagonalize an Hermitian matrix H through a unitary matrix U :

$$H = U\Lambda U^* \quad , \quad U^* = U^{-1}. \quad (2.12)$$

Then one can write the initial measure $P(H)dH$ in terms of the coefficients of Λ and U . Clearly the measure on the unitary group must factorize from the eigenvalues measure since $P(H)$ is invariant through action of the unitary group. Let us explain by a simple argument the well known result

$$P(H)dH = d\mu(U)e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N d\lambda_i. \quad (2.13)$$

To understand the appearance of the non-trivial Vandermonde factor $\prod_{i<j} (\lambda_i - \lambda_j)^2$ (in addition to the ordinary trivial factor $P(H) = e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2}$) we need only to compute the Jacobian at origin from the H variables to the λ variables and the variables parameterizing U near the origin. For this purpose, we can derive the relation $U^*U = 1$ with respect to a set of local parameters U_r for a local chart of the unitary group near the origin. This gives

$$S_r = U^* \frac{\partial U}{\partial U_r} = -S_r^*, \quad (2.14)$$

the tangent space to the unitary group at the origin being the anti-hermitian matrices. From $H = U\Lambda U^*$ one finds

$$\frac{\partial H}{\partial U_r} = \frac{\partial U}{\partial U_r} \Lambda U^* + U \Lambda \frac{\partial U^*}{\partial U_r}, \quad (2.15)$$

hence

$$(U^* \frac{\partial H}{\partial U_r} U)_{ij} = (S_r \Lambda - \Lambda S_r)_{ij} = (S_r)_{ij} (\lambda_j - \lambda_i). \quad (2.16)$$

Furthermore

$$U^* \frac{\partial H_{ij}}{\partial \lambda_k} U = \frac{\partial \Lambda_{ij}}{\partial \lambda_k} = \delta_{ij} \delta_{ik}, \quad (2.17)$$

so that the Jacobian to compute is

$$J = \left| \begin{array}{cc} \frac{\partial H_{ii}}{\partial \lambda_k} & \frac{\partial H_{ij}}{\partial \lambda_k} \\ \frac{\partial H_{ji}}{\partial U_r} & \frac{\partial H_{ij}}{\partial U_r} \end{array} \right| = \left| \begin{array}{cc} 1 & 0 \\ f(\lambda, U) & (\lambda_j - \lambda_i) S_r(U) \end{array} \right| = g(U) \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad (2.18)$$

Clearly the presence of this Vandermonde determinant means that the eigenvalues of a random matrix in the GUE case are not independent, but repel each other since the measure vanishes at coinciding eigenvalues. Physically this level repulsion is analogous to some kind of Pauli exclusion principle between eigenvalues, or to some two body logarithmic interaction:

$$e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N d\lambda_i = e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2 + 2\sum_{i<j} \log |\lambda_i - \lambda_j|} \prod_{i=1}^N d\lambda_i, \quad (2.19)$$

which is analogous to Coulomb repulsion in two dimensions (also logarithmic).

It is possible to use the theory of orthogonal polynomials to analyze the large N limit and recover Wigner's law for this system or for more complicated non-Gaussian measures on H (for the GUE, orthogonal polynomials are simply Hermite polynomials). This is e.g. done in [4]. See also [5] for another reference book on the subject.

In this lecture we prefer to stress the supersymmetric approach to this problem. It makes particularly transparent how Wigner's law results from a mean-field theory and a saddle point expansion which expresses the subtle balance between the Gaussian and Vandermonde terms in 2.19.

2.2 Supermathematics

In this presentation we follow the excellent concise review by Mirlin [6].

Grassmann or anticommuting or Fermionic variables are pairs of independent variables, which for convenience are noted as complex conjugates, $\chi_1, \dots, \chi_N, \chi_1^*, \dots, \chi_N^*$ with the following properties

$$\chi_i \chi_j = -\chi_j \chi_i, \quad \chi_i^* \chi_j = -\chi_j \chi_i^*, \quad \chi_i^* \chi_j^* = -\chi_j^* \chi_i^*, \quad (2.20)$$

$$\int d\chi_i = \int d\chi_i^* = 0, \quad \int \chi_i d\chi_i = \int \chi_i^* d\chi_i^* = \frac{1}{\sqrt{2\pi}}. \quad (2.21)$$

Conjugation for Fermionic variables is not involutive but antiinvolutive, and it does not reverse the ordering of a product: :

$$(\chi^*)^* = -\chi, \quad (\chi\psi)^* = \chi^*\psi^*. \quad (2.22)$$

Then for any $N \times N$ matrix M we find using these rules that

$$\int \prod_i d\chi_i^* d\chi_i e^{-\chi^* M \chi} = \det\left(\frac{M}{2\pi}\right), \quad (2.23)$$

where χ is now considered a vector (χ_1, \dots, χ_N) . For ordinary, also called *bosonic*, complex conjugate variables S, S^* we would have got instead, requiring $\Re M > 0$ for convergence:

$$\int \prod_i dS_i^* dS_i e^{-S^* M S} = \det^{-1}\left(\frac{M}{2\pi}\right). \quad (2.24)$$

Supersymmetric expressions are obtained by assembling bosonic and Fermionic variables in a symmetric way. For instance a supervector is

$$\Phi = (S_1, \dots, S_N, \chi_1, \dots, \chi_N), \quad \Phi^* = (S_1^*, \dots, S_N^*, \chi_1^*, \dots, \chi_N^*). \quad (2.25)$$

A supermatrix is $M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$, $2N$ by $2N$, in which a and b are ordinary bosonic and σ, ρ are anticommuting variables. We use Latin indices from 1 to N

and Greek indices with values b and f to distinguish the bosonic and Fermionic parts, so that we write

$$M = M_{\alpha\beta,ij} , \quad M_{bb,ij} = a_{ij} , \quad M_{ff,ij} = b_{ij} , \quad M_{bf,ij} = \sigma_{ij} , \quad M_{fb,ij} = \rho_{ij} . \quad (2.26)$$

Traces and determinants generalize into supertraces and superdeterminants:

$$\text{Str} M = \text{Tra} - \text{Tr} b \quad , \quad \text{Sdet} M = \det(a - \sigma b^{-1} \rho) \cdot \det b^{-1} . \quad (2.27)$$

We still have

$$\text{Str} \log M = \log \text{Sdet} M . \quad (2.28)$$

Furthermore for any supermatrix (requiring $\Re M_{bb} > 0$ for convergence):

$$\int d\Phi^* d\Phi e^{-\Phi^* M \Phi} = \text{Sdet} M^{-1} . \quad (2.29)$$

Remark that for a supersymmetric supermatrix M , i.e. one in which $\rho = \sigma = 0$ and $a = b$, this integral is normalized, namely $\text{Sdet} M = 1$.

The inverse of a supermatrix can be computed as

$$M^{-1} = \begin{pmatrix} (a - \sigma b^{-1} \rho)^{-1} & -(a - \sigma b^{-1} \rho)^{-1} \sigma b^{-1} \\ -b^{-1} \rho (a - \sigma b^{-1} \rho)^{-1} & b^{-1} (1 + \rho (a - \sigma b^{-1} \rho)^{-1} \sigma b^{-1}) \end{pmatrix} . \quad (2.30)$$

Like for an ordinary matrix we can define a resolvent, or two point function

$$\int d\Phi^* d\Phi \Phi_{\alpha i} \Phi_{\beta j}^* e^{-\Phi^* M \Phi} = (M^{-1})_{ij}^{\alpha\beta} \text{Sdet} M^{-1} . \quad (2.31)$$

Further references on supercalculus, supermanifolds can be found in [7].

2.3 Wigner's law

Let us return to the proof of the Wigner's law using the supersymmetric formalism. The advantage of supersymmetry is to allow the integrated resolvent to be written as a functional integral in an ordinary field theory because the corresponding normalizing determinant is 1. More precisely let us return to

$$\langle \nu(E) \rangle = - \lim_{\varepsilon \rightarrow 0_+} \int P(H) dH \frac{1}{\pi N} \Im \text{Tr} \frac{1}{E - H + i\varepsilon} . \quad (2.32)$$

We would have indeed with ordinary variables

$$\langle (E - H + i\varepsilon)_{ij}^{-1} \rangle = -i \int P(H) dH \frac{\int S_i^* S_j e^{iS^*(E-H+i\varepsilon)S} dS^* dS}{\int e^{iS^*(E-H+i\varepsilon)S} dS^* dS} \quad (2.33)$$

where the imaginary sign has been chosen so that the bosonic part of the integral converges, using the small positive part $e^{-\varepsilon S^* S}$. The integral of such a quotient is not easy to manipulate. But with supervariables taking the self-normalization into account we can write

$$(E - H + i\varepsilon)_{ij}^{-1} = -i \int S_i^* S_j e^{i\Phi^*(E-H+i\varepsilon)\Phi} d\Phi^* d\Phi \quad (2.34)$$

so that integration over H can be performed explicitly. Indeed

$$\langle (E - H + i\varepsilon)_{ij}^{-1} \rangle = -i \int P(H) dH \int S_i^* S_j e^{i\Phi^*(E-H+i\varepsilon)\Phi} d\Phi^* d\Phi \quad (2.35)$$

and the Gaussian integral of an exponential linear in H can be performed exactly, giving rise to an exponential quartic in Φ , something which physicists call a " Φ^4 interaction":

$$\langle e^{i \sum_{ij} \Phi_i^* H_{ij} \Phi_j} \rangle = \exp \left\{ -(1/2N) \sum_{ij} (\Phi_i^* \Phi_j)(\Phi_j^* \Phi_i) \right\}. \quad (2.36)$$

The essential step is now to recast this quartic term in the form of a so-called "vector model", that is to factorize the quartic term as a square. Taking carefully into account the anticommutation rules one finds:

$$\sum_{ij} (\Phi_i^* \Phi_j)(\Phi_j^* \Phi_i) = \left(\sum_i S_i^* S_i \right)^2 + 2 \left(\sum_i S_i \chi_i^* \right) \left(\sum_i S_i^* \chi_i \right) - \left(\sum_i \chi_i^* \chi_i \right)^2. \quad (2.37)$$

When rewritten this way, one can introduce a representation of the quartic term as an integral over a single supermatrix field R . Physicist call this idea a "Hubbard-Stratonovich" transformation. In practice the many variables H have been reduced to four "mean field" variables, the coefficients of R . We see that the key fact under this phenomenon is the independence of the variables H_{ij} ¹.

More precisely we have

$$\exp \left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^* \Phi_j)(\Phi_j^* \Phi_i) \right\} = \int dR \exp \left\{ -\frac{N}{2} \text{Str} R^2 - i \sum_i \Phi_i^* R \Phi_i \right\} \quad (2.38)$$

with

$$R = \begin{pmatrix} a & \rho^* \\ \rho & ib \end{pmatrix}. \quad (2.39)$$

Indeed due to the i factor in the fermion-fermion b term, the bosonic part of the quadratic form $\text{Str} R^2 = a^2 + b^2 + 2\rho^* \rho$ is positively definite, ensuring convergence.

We can now interchange the Φ and R integrations and perform the Φ inte-

¹For a random matrix whose coefficients are not independent, this mean-field "Hubbard-Stratonovich" transformation is not valid and Wigner's law may no longer hold; however if a random matrix has for instance $N^2/p(N)$ independent coefficients, each spanning an orbit of $p(N)$ sites in the matrix, we may search for a reduced representation with only $O(p(N))$ "Hubbard-Stratonovich" variables and still expect Wigner's law for instance if $p(N)/N \rightarrow 0$ as $N \rightarrow \infty$, see below.

grations which are now quadratic:

$$\begin{aligned}
\langle \text{Tr}(E - H + i\varepsilon)^{-1} \rangle &= -i \int dR \int d\Phi^* d\Phi \sum_k S_k^* S_k \\
&\quad \exp\left\{i(E + i\varepsilon) \sum_i \Phi_i^* \Phi_i - \frac{N}{2} \text{Str} R^2 - i \sum_i \Phi_i^* R \Phi_i\right\} \\
&= N \int dR (E - R + i\varepsilon)_{bb}^{-1} \\
&\quad \exp\left\{-\frac{N}{2} \text{Str} R^2 - N \text{Str} \log(E - R + i\varepsilon)\right\}, \quad (2.40)
\end{aligned}$$

so that

$$\begin{aligned}
\langle \nu(E) \rangle &= - \lim_{\varepsilon \rightarrow 0_+} \frac{1}{\pi} \Im \int dR (E - R + i\varepsilon)_{bb}^{-1} \\
&\quad \exp\left\{-\frac{N}{2} \text{Str} R^2 - N \text{Str} \log(E - R + i\varepsilon)\right\}. \quad (2.41)
\end{aligned}$$

This R integral as $N \rightarrow \infty$ should be accessible to saddle point analysis, since the integrand is of the form $e^{-NS(R)}$ with N large in front of the action $S(R) = \frac{1}{2} \text{Str} R^2 + \text{Str} \log(E - R + i\varepsilon)$. The saddle points satisfy in the limit $\varepsilon \rightarrow 0$

$$R = (E - R)^{-1}; \quad R^2 - ER + 1 = 0 \quad (2.42)$$

so the corresponding eigenvalues are

$$\mathcal{E} = E/2 - i\sqrt{1 - E^2/4}, \quad \mathcal{E}^* = E/2 + i\sqrt{1 - E^2/4}, \quad (2.43)$$

with $\mathcal{E}\mathcal{E}^* = 1$, $E - \mathcal{E} = \mathcal{E}^*$. The corresponding bosonic saddle points are

$$R = E/2 - i\sqrt{1 - E^2/4} \begin{pmatrix} s_a & 0 \\ 0 & s_b \end{pmatrix}. \quad (2.44)$$

with $s_a = \pm 1, s_b = \pm 1$. These saddle points are not along the original path of integration, which is a real and b real. So we have to shift the a integration contour to include an imaginary part $-is_a\sqrt{1 - E^2/4}$ and the b contour to include an imaginary part $-iE/2$. We remark that in order to avoid crossing singularities we must take into account the positive sign of ε and choose $s_a = +1$ if $\varepsilon > 0$. So we are left with two saddle points at $s_b = \pm 1$. But it is possible to explicitly perform the Fermionic integration and check that the saddle point at $s_b = 1$ dominates the saddle point at $s_b = -1$, which is suppressed by a $1/N$ factor.

Indeed we have

$$(E - R)_{bb}^{-1} = \frac{1}{E - a} [1 - \rho^* \rho \frac{1}{(E - a)(E - ib)}]^{-1}, \quad (2.45)$$

$$\text{Sdet}(E - R) = \frac{E - a}{E - ib} [1 - \rho^* \rho \frac{1}{(E - a)(E - ib)}]. \quad (2.46)$$

Hence

$$\begin{aligned} \langle \nu(E) \rangle &= -\frac{1}{\pi} \Im \int da db d\rho^* d\rho e^{-N/2(a^2+b^2+2\rho^*\rho)} \\ &\quad \frac{(E-ib)^N}{(E-a)^{N+1}} [1 - \rho^* \rho \frac{1}{(E-a)(E-ib)}]^{-N-1} , \end{aligned} \quad (2.47)$$

so that, performing exactly the ρ and ρ^* integration, and translating the contours, we get, remembering that by (2.21) and anticommutation $\int d\rho^* d\rho \rho^* \rho = -\frac{1}{2\pi}$:

$$\begin{aligned} \langle \nu(E) \rangle &= \frac{1}{2\pi^2} \Im \int_{\Im a = -\sqrt{1-E^2/4}} da \int_{\Im b = -E/2} db e^{-N/2(a^2+b^2)} \\ &\quad \frac{(E-ib)^N}{(E-a)^{N+1}} N[-1 + \frac{1}{(E-a)(E-ib)} + O(1/N)] . \end{aligned} \quad (2.48)$$

At the vicinity of the first saddle point we have $a = \mathcal{E} + \delta a$, $ib = \mathcal{E} + i\delta b$ and by ordinary Hessian analysis we need to expand to second order in δa and δb . We find

$$e^{-N/2(a^2+b^2)} \frac{(E-ib)^N}{(E-a)^N} = e^{-N/2(1-\mathcal{E}^2)(\delta a^2+\delta b^2)+O(\delta^3)} . \quad (2.49)$$

Furthermore at the first saddle point we have $[-1 + \frac{1}{(E-a)(E-ib)}] = -1 + \mathcal{E}^2$ and $\frac{1}{(E-a)} = \mathcal{E}$ so that the Hessian approximation near the first saddle point gives a contribution

$$\begin{aligned} \langle \nu_1(E) \rangle &= \frac{1}{2\pi^2} \Im \int \delta a \int \delta b e^{-N/2(1-\mathcal{E}^2)(\delta a^2+\delta b^2)} \mathcal{E}(-1 + \mathcal{E}^2 + O(1/N)) \\ &= \frac{1}{\pi} \sqrt{1-E^2/4} + O(1/N) . \end{aligned} \quad (2.50)$$

At the second saddle point we have $a = \mathcal{E} + \delta a$, $ib = \mathcal{E}^* + i\delta b$ so that $-1 + \frac{1}{(E-a)(E-ib)} = 0$ at this other saddle point. Therefore to leading order as $N \rightarrow \infty$ there is no contribution to $\nu(E)$ from this other saddle point. It is possible to bound the contributions away from the saddles and to conclude therefore

$$\langle \nu(E) \rangle = \begin{cases} \frac{1}{\pi} \sqrt{1-E^2/4} & \text{for } |E| \leq 2 , \\ 0 & \text{otherwise .} \end{cases} \quad (2.51)$$

The computation of level correlations is similar but complicated by lack of positivity of the Hubbard-Stratonovich form, which leads to technical complications. There is also an additional symmetry, and the main non-trivial mean field integral therefore takes values in a coset superspace parameterized by eight real variables, four bosonic and four Fermionic [6].

Let us also stress that the corrections as $N \rightarrow \infty$ to Wigner's law can be in principle systematically computed but the computation becomes more and more difficult in practice for higher order terms. However it is relatively easy

to derive e.g. crude bounds on the probability that a random $N \times N$ matrix develops a large norm using some kind of Tchebycheff inequalities. For instance it is an instructive exercise to prove [8]

Lemma 1 *For sufficiently large a and N one has*

$$\text{Prob}(\|H\| \geq a\sqrt{6}) \leq 8Na^6/27e^{-N^{1/3}a^2/3} . \quad (2.52)$$

The Tchebycheff inequality here is simply to use

$$\text{Prob}(\|H\| \geq a\sqrt{6}) \leq (6a^2)^{-m} < \text{Tr}H^{2m} > \quad (2.53)$$

and to optimize over m . Gaussian integration for $< \text{Tr}H^{2m} >$ can be explicitly performed through Wick's theorem. It leads to the evaluation of so called Feynman graphs. A careful counting of these graphs leads to the proof of the lemma by taking m proportional to a^2 . This bound is not optimal but shows clearly that the probability for a random matrix normalized in this way to develop eigenvalues much larger than $O(1)$ is very small. Such crude bounds are sufficient for complete control of the Anderson model in the regime $|p^2 - E| \gg \lambda^2$, see below.

3 The Anderson Model

The Anderson model of an electron in a random potential corresponds to the Hamiltonian

$$H = -\Delta + \lambda V(x),$$

acting on the Hilbert space $L^2(\mathbf{R}^2)$, where Δ is the usual Laplacian and V is a real Gaussian process on \mathbf{R}^2 with short range correlations. λ is a coupling constant that allows to adjust the strength of the random disorder with respect to the deterministic part.

This model was initially proposed to describe the electron motion in doped semi-conductors at low temperature or in normal disordered metals. It is now the central model for the theory of electronic transport and wave propagation in disordered systems [9]. It was conjectured by Anderson as soon as 1958 [10] that such a model exhibits a localized phase in which the electrons are trapped by the defects. In 1979 it was argued that this model has a phase transition in dimensions three or more between the localized phase and an extended one [11].

The localized phase is now well under control. In one dimension, localization was rigorously established for any disorder at the end of the seventies [12, 13]. Later localization was established in any dimension at strong disorder or for energies out of the conduction bands [14]. A simplified and more efficient method to get this result was given in [15].

In contrast the weak disorder regime is still poorly understood. In two dimensions it has been argued [11, 16] and numerically established [17] that localization persists at arbitrarily small disorder, with a localization length diverging like $O(e^{c/\lambda^2})$. In dimension three, numerical simulations confirm the existence

of the Anderson transition [17], leading to an extended phase. In addition, analytical results [18] and other numerical calculations show that the level spacing distribution follows the Wigner-Dyson distribution for random matrix theory (RMT) [19]. This gave the motivation for a description of mesoscopic system in terms of RMT [20]. This method has been very successful when compared to experiments and was the source of developments of supersymmetric methods [21, 6] in solid state physics. But this heuristic connection between random matrices and the Anderson model remained mysterious. And on the rigorous level it is still a mathematical challenge up to now to prove even regularity of the DOS at weak disorder in the conduction band (that is, analyticity in energy in a narrow band around a real interval in the conduction band of the deterministic Hamiltonian, see Conjecture 1 below). This is still unknown in either dimensions two or three (see [22] for recent continuity results outside the conduction band).

Using a phase-space analysis inspired by the renormalization group method around Fermi surfaces in condensed matter [23, 24, 25], G. Poitot and coauthors [26] have understood the connection between the Anderson model and random matrices. They established that the effective Hamiltonian near the Fermi level is given indeed by a random matrix model. In the simplest case, that of two dimensions this random matrix model is almost identical to the GUE (Gaussian Unitary Ensemble), but just contains one extra discrete symmetry. This symmetry is called the Flip Symmetry in [1], and the corresponding Flip Matrix model has been analyzed through the supersymmetric method. In three dimensions the flip symmetry becomes a continuous $U(1)$ symmetry, and produces more complicated correlations between matrix elements [26]; however even in that case, these correlations are not expected to change the main statistical properties of the spectrum, such as Wigner's law for the density of states.

In the coming subsections we will summarize the content of [1]. Let us recall that in [1] the flip symmetry was slightly simplified. This flip symmetry is indeed essentially a \mathbb{Z}_2 symmetry, but which effectively degenerates into a larger symmetry near the diagonals of the matrix, which correspond to degenerate rhombuses in the momentum representation. Therefore in the last section of this review we will refine the phase space analysis of the Anderson model and propose improved matrix models to study this slight degeneracy near the diagonals of the flip symmetry, using some hierarchical approximations that mimic the true situation but are more tractable from the point of view of the supersymmetric mean field analysis. Similar ideas could also apply to the study of the $U(1)$ three dimensional symmetry.

3.1 Phase Space Analysis

As in other condensed matter models, the ultraviolet region in momentum space is irrelevant and cut off in the Anderson model. The most common way to do this is to replace ordinary space by a lattice such as \mathbb{Z}^d . This corresponds physically to the so-called tight binding approximation in which the electron spends all its time on lattice sites in a crystal, jumping from one site to an other. This lattice

model automatically cuts off large momenta since the Fourier transform of the lattice \mathbb{Z}^d is simply the torus $[-\pi, \pi]^d$.

Since the lattice cutoff breaks rotation invariance, it is also common to consider a rotation invariant momentum cut off. For that we consider the free Hamiltonian without the random potential, which is simply the Laplacian in a suitable unit system:

$$H_0 = -\Delta \quad (3.54)$$

and the free retarded Green's function:

$$G_0 = \frac{1}{H_0 - E - i\varepsilon}. \quad (3.55)$$

also called the *propagator* in physics. In Fourier space this is a diagonal multiplication operator. A rotation invariant momentum cutoff can be implemented by multiplying this operator by a smooth C^∞ function κ with compact support which suppresses large momenta. For instance we can ask that this function is identically 1 for $|p| \leq A$ and 0 for $|p| \geq A + 1$, where A is a fixed number, e.g. 10. This leads to define the cutoff free retarded Green's function:

$$G_{0,\kappa}(p) = \frac{\kappa(p)}{p^2 - E - i\varepsilon}. \quad (3.56)$$

This is usually called in physics the "jellium" cutoff, and we call the corresponding model the jellium Anderson model. Regular perturbation theory for this model consists in computing the interacting Green's function as a power series in λ through a resolvent expansion:

$$G = G_0 \frac{1}{1 + \lambda V G_0} = G_0 - G_0 \lambda V G_0 + G_0 \lambda V G_0 \lambda V G_0 + \dots \quad (3.57)$$

The rotation invariant cutoff consists in replacing everywhere G_0 by $G_{0,\kappa}$. Remark that the convergence of the power series for fixed V is no problem at λ small enough, since this is a geometric series. However we are not interested in any particular V but on the average over V . The corresponding integrated Green's function is obtained by integrating over V , using the Gaussian rules of integrations, called Wick's theorem in physics. Only terms with even numbers of V survive and the result can be indexed by drawings called Feynman graphs. In these Feynman graphs, the G_0 propagators can be pictured as full lines, whether the covariances for contracted V , which are δ functions in x -space, are usually represented by dotted lines (see Figure 1). In this case the first non trivial graph also called tadpole graph (see Figure 2) gives a momentum space contribution

$$C = \lambda^2 \int d^d p \frac{\kappa(p)}{p^2 - E - i\varepsilon}. \quad (3.58)$$

This is a pure number, with a finite imaginary part δ as $\varepsilon \rightarrow 0$. For instance in two dimensions and for $E = 1$, this imaginary part δ is $\lambda^2 \pi^2$. Therefore physicists, resumming only the graphs G_n with n consecutive tadpoles (see Figure 3), which form a geometric series, and neglecting all other graphs, can

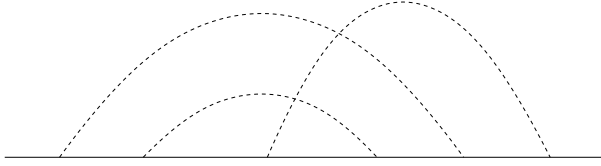


Figure 1: A typical Feynman graph

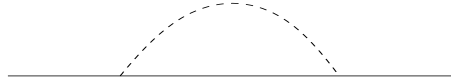


Figure 2: The first non trivial graph, the "tadpole"

perform the limit $\varepsilon \rightarrow 0$:

$$G_{tadpole} = G_0 \sum_{n=0}^{\infty} (\delta G_0)^n = \frac{\kappa(p)}{p^2 - E - i\delta\kappa(p)}. \quad (3.59)$$

With their usual boldness, they conclude that the averaged Green's function should be regular in momentum space, and consequently also decay in x -space at a spatial scale inverse of δ , at least for small λ . The corresponding conjectures, alas unproved yet, could be described as follows:

Conjecture 1: Analyticity of the Averaged Density of States

Let $[a, b]$ is a fixed interval well inside the spectrum of the unperturbed Laplacian spectrum (this means $[a, b] \subset]0, 2d[$ for the lattice model, and $[a, b] \subset]0, A[$ for the jellium model). For any λ small, the averaged density of states $\langle \nu(E) \rangle$ is analytic in E in a rectangle centered on the real interval $[a, b]$ and of width $O(\lambda^2)$, hence of type $[a, b] \times [-iO(\lambda^2), +iO(\lambda^2)]$.

Conjecture 2: Scaled Polynomial Decay of the Averaged Green's Function

There exists a constant c such that for any integer q , there exists a constant K_q such that the averaged Green's function $\langle G \rangle$ in direct space obeys long range spatial decay of power q and scale $O(\lambda^2)$:

$$| \langle G(x, y) \rangle | \leq K_q (1 + c\lambda^2 |x - y|)^{-q}. \quad (3.60)$$

These conjectures are not completely optimal, but even in this form they are still a challenge. It is in fact expected that we have analyticity in a rectangle

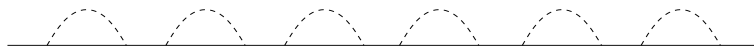


Figure 3: The graph G_n with n tadpoles

with $[a, b]$ closer and closer to the full unperturbed spectrum as $\lambda \rightarrow 0$. Also in the case of the lattice model one expects in fact exponential scaled decay of order λ^2 . Since a compact support cutoff κ cannot be analytic, its Fourier transform does not decay exponentially. So for the jellium model one expects at most fractional exponential decay (if the function κ is Gevrey).

Perhaps more interesting than the conjectures themselves is what they mean: namely that no matter how small the interaction, it regularizes the average Green's function. This is somewhat similar the phenomenon of dynamic mass generation in field theories like QCD, although here the regulator is polynomial in λ .

These conjectures would be important also to justify perturbation theory and the tadpole approximation in the small coupling regime of the Anderson model. As is well known the ordinary perturbative approach is plagued by mathematical problems. Indeed due to the large number of Feynman graphs (there are $n!! = (n-1)(n-3)\dots$ graphs at order n) the integrated perturbation series has zero radius of convergence. Over the years, techniques have been developed to overcome this problem by replacing the complete divergent Feynman series by carefully truncated expansions with Taylor remainders that can be rigorously bounded. This set of techniques goes under the name of "constructive field theory" [27].

Constructive field theory tells us that in order to achieve non-perturbative theorems in this kind of situations where the interplay between propagation and interaction is non-trivial, the best frame is to analyze the theory in phase-space. Some multi-scale *renormalization group* analysis is usually necessary. Renormalization group is the tool which computes long range behavior from local physical interactions. Cluster and Mayer expansions are the correct non-perturbative steps which make the renormalization group mathematically well-defined [27]. The guiding line is to identify the relevant degrees of freedom of the problem in phase space, and then to perform some carefully controlled perturbation theory which roughly derives at most one perturbation step for each such degree of freedom.

A Hamiltonian, whether random or not, acts indeed on a Hilbert space of quantum states. Using a system of units in which $\hbar = 1$ a quantum *degree of freedom* corresponds roughly to a unit volume in phase space. Although the quantum Hilbert space for a particular problem may be formally infinite dimensional (such as $L^2(\mathbb{R})\dots$), in practice any physical problem should in fact be well approximated by a finite dimensional version of this Hilbert space. The main problem is to identify the relevant degrees of freedom of the system, and to control their proliferation in the ultraviolet (field theory) or infrared (statistical mechanics) limit.

A partition of the classical phase space into cells of unit volume may be viewed as giving a particular basis of this finite dimensional Hilbert space. Not all partitions are admissible; they should roughly respect the Heisenberg uncertainty principle ultimately related to the symplectic geometry of phase space. Any product of the typical dimensions of any cell in the directions of a conjugate pair of positions and momenta should not be much smaller than 1.

Some decompositions may be more convenient than others for particular problems. In the lattice Anderson model on a lattice, the phase space is $\mathbb{Z}^d \times [0, 2\pi]^d$ since positions lie on a square lattice and momenta lie in the dual tori. In the jellium Anderson model, it is $\mathbb{R}^d \times B(0, A + 1)$, where the closed ball $B(0, A + 1)$ in \mathbb{R}^d with center 0 and radius $A + 1$ is the compact support of the cutoff function κ . This is not very different from the previous case since κ restricts the large momenta, and the x -space should be again well approximated by a lattice of cubes of side size roughly the inverse of A , hence of order $O(1)$. Indeed all propagators and functions built out of them will be roughly constant within each of these cubes.

The initial natural partitioning of such a phase space is to associate to each lattice site or cube in direct x -space a *degree of freedom*. Momentum space is compact, and the only reason for which the problem has infinite number of degrees of freedom is the infrared problem, or thermodynamic limit: we want to understand the theory in infinite volume in x -space.

This obvious partitioning is well adapted to the study of the localized phase of the Anderson model, in which the random potential dominates over the propagation. Indeed a random identically distributed potential V is simply a multiplication operator in x -space. It is a diagonal operator in x space with i.i.d. diagonal coefficients $V(x)$ for $x \in \mathbb{Z}^d$. Spectrum is pure point and the eigenvalues of this operator are obviously the wave functions concentrated on single sites, or δ functions in x space. The density of states and statistics of eigenvalues are Poissonian: the presence of an eigenvalue at some energy E simply means that for some x we have $V(x) = E$, and this information has no influence on the values of $V(y)$ for $y \neq x$. Therefore in this case $R_2(s) \simeq_{s \rightarrow 0} \delta(s) + 1$.

Most of the rigorous work on the Anderson model is concerned with the localized phase. In this phase the addition of a hopping term such as a lattice Laplacian is a small perturbation which does not fundamentally modify the spectrum, which remains pure point, and the localized character of the eigenfunctions.

But this partitioning of phase space is not the right one to study the other regime in which it is the lattice Laplacian with its continuous spectrum which is the main effect, and the random potential which should be the perturbation.

From now on let us restrict to a single averaged Green's function or the density of states, leaving the level correlations for future studies. To prove analyticity in the rectangle of Conjecture 1, we need only to consider a fixed value E_0 in $[a, b]$ and, translating E to $E_0 + E$, to prove analyticity for $|E| \leq O(\lambda^2)$. If we can do this for any $E_0 \in [a, b]$, Conjecture 1 follows. Since this will not change anything except a global rescaling we always choose $E_0 = 1$ in what follows.

If we take seriously the propagator as a guide to the right partitioning of phase space, we recognize that the interesting region in momentum space is the singular region $p^2 = 1$. This is the d -dimensional sphere, for instance in two dimensions it is the circle of radius 1. We know from the theory of renormalization group around extended singularities [24, 25] that in this case we should perform a change of basis, dividing the region $p^2 \simeq 1$ into smaller and

smaller cells to analyze it with great care; correspondingly x space should be cut into dual boxes that will be larger and larger as we approach the singularity, hence correspond to longer and longer distance effects.

This decomposition correspond to cut first momentum space into shells according to a geometric progression which pinches more and more the singularity. Fixing a rate $M > 1$ for this geometric progression (for instance $M = 2$), this means that we write

$$\kappa(p) = \sum_{j=0}^{j_{max}} \kappa_j(p) \quad (3.61)$$

where $\kappa_j(p)$ is a smooth function with compact support which roughly ensures $|p^2 - 1| \simeq M^{-j}$.

The index j plays the role of a renormalization group index, and the corresponding region in momentum space is called the j -th slice or momentum shell. The last index j_{max} is chosen so that $M^{-j_{max}} \simeq \delta = \pi^2 \lambda^2$. Indeed the expansion should not be infinite. The last function $\kappa_{j_{max}}(p)$ for the last slice j_{max} should be different, meaning only $|p^2 - 1| \leq M^{-j_{max}}$. The reason is that we expect the basic physical intuition behind (3.59) to be correct. Averaged Green's functions should decay after that scale; equivalently the singularity of the free propagator should be screened by the appearance of an imaginary part due to the interaction with the random potential.

This screening phenomenon, like mass generation under influence of interaction in field theory, can only occur at a scale where interaction effects become of the same size than the free propagator. The conclusion is that we must distinguish two regimes:

- The regime $|p^2 - 1| \gg \lambda^2$, or $j \leq j_{max}$. In this region the free propagator should dominate over the interaction and regular cluster and Mayer expansion should prove that this part of the Green's function remains in every respect close to the free one.

- The regime $|p^2 - 1| \gg \lambda^2$ or $j \simeq j_{max}$, which corresponds to a few renormalization group slices (or in fact just one if we take M big enough). In this last slice clearly ordinary perturbation theory (even recast under the form of cluster or Mayer expansion) cannot work since propagators and typical random potentials are of the same size (otherwise the non-trivial effect of screening would be impossible).

In this regime the functional integral oscillates wildly. A first attempt to exploit the corresponding cancellations through Ward identities was made in [30]. However this turned out to be a difficult and not completely conclusive approach. Oscillating integrals with analyticity properties can often be better analyzed by finding saddle points and shifting contours to pass through them. In favorable cases the leading part of the integral then comes simply from a Hessian approximation near a single dominant saddle. This saddle becomes a new expansion point for the theory, similar to a change of vacuum in field theory. The corresponding shift is definitely a non-perturbative tool, very difficult to understand in terms of combinations of the initial Feynman graphs of Figure 1.

Such a shift precisely occurs in the supersymmetric formalism of the previous section: the contour for the mean fields a and b has been shifted to pass through a non trivial saddle point that lied out of the original integration contour. In this way a highly oscillating integral, has been replaced by an equal one which is much simpler to analyze. We must explain now why the Anderson model is analogous to a matrix model, so that the same strategy should also work there.

3.2 Sectors and the Random Matrix Approximation

Momentum slices or shells without further decomposition are not suited for a phase space analysis. Indeed a free propagator restricted to such a shell has no simple dual decay properties in direct space. The correct phase space analysis must split further each shell into angular sectors, along the directions tangential to the extended singularity. What limits the length of the sectors in these directions is the curvature of the singularity [25]; (see also [28] for a non-spherical example). In the case of a spherical singularity such as the one of the jellium Anderson model, there are two natural and acceptable solutions to the splitting into sectors:

- split the spherical shell of width M^{-j} into roughly $M^{(d-1)j}$ sectors (noted by Greek letters such as α or σ) with same size in all directions. This is the most natural idea and the corresponding sectors are called isotropic. It is however not optimal for momentum conservation analysis. The direct space is then cut into a lattice \mathcal{D}_j of cubic boxes Δ of side size M^j for each value of j .
- split the spherical shell of width M^{-j} into roughly $M^{(d-1)j/2}$ sectors, still noted σ , of length $M^{-j/2}$ in all directions tangent to the singularity. These sectors are called anisotropic. This is the optimal tool to analyze e.g. momentum conservation, but the price to pay is that the dual decomposition of x -space gives rise to anisotropic parallelepipeds which change as the sector changes. For each sector σ in a shell j we have a different lattice \mathcal{D}_σ of spatial boxes covering \mathbb{R}^d , which have length M^j in the direction of σ , and length only $M^{j/2}$ in the other orthogonal directions (since for the sphere these orthogonal directions correspond to the tangent directions at σ).

In the simple isotropic case, a phase space cell of roughly unit volume is therefore made of a value of j in $[0, j_{max}(\lambda)]$, a spatial box $\Delta \in \mathcal{D}_j$ and a momentum cell σ in the j -th shell. The true Hilbert space of the theory is well approximated by a finite dimensional Hilbert space whose basis is indexed by these phase space cells.

Now at last comes the analogy with random matrices. In x space, V is a real diagonal operator; therefore in p space $V(p)$ is complex with

$$V(-p) = \bar{V}(p), \quad (3.62)$$

and V is a random convolution operator $Vf(p) = \int dq V(q)f(p+q)$ with covariance $\langle V(p)\bar{V}(q) \rangle = \delta(p-q)$. A delta or short range covariance in x space translates into a white noise in p space, that is all momenta correspond to independent variables with a priori equal weight. If we use isotropic sectors to

simplify, it is natural to discretize the momentum space for V with cells of the same dimensions that the angular sectors used for the propagator. When two propagators with momenta α and β join at a potential V , the momentum of V , or "transfer momentum" by overall translation invariance of our theory, is $p = \alpha - \beta$. There is therefore an effective ultraviolet cutoff on V (because α and β being bounded, so is $\alpha - \beta$).

Here is a first subtlety. There is no reason for α and β to lie in the same shell. The operator V can be decomposed into the shell-conserving or diagonal part $V_s = \sum_j V_{s,j}$ and the shell-changing or off-diagonal part V_o . For the time being, let us neglect V_o and consider an operator $V_{s,j}$ acting in a single shell.

The covariance for V has no singularity and we should therefore discretize the momenta for V into roughly M^{dj} cells τ of side size M^{-j} . Hence if we divide space according to the lattice \mathcal{D}_j , to a given cube Δ should correspond roughly M^{dj} discrete variables $V_{\tau,\Delta}$, which are roughly independent and identically distributed. The interaction being local in x space, such a variable $V_{\tau,\Delta}$ must join sectors (α, Δ) and (β, Δ) spatially localized in the same cube Δ of \mathcal{D}_j . From now on fixing such a cube Δ , we can therefore consider the variables $V_{\tau,\Delta}$, as the discrete elements of a random matrix that we call $V(\Delta)$.

Let α and β be two indices parametrising two sectors of the same shell j . We need to know which pairs of sectors a given independent variable $V_{\tau,\Delta}$ can join, so to analyze the relation $\tau = \alpha - \beta$. The answer happens to be strongly dimension-dependent.

A first remark is that if $\tau = \alpha - \beta$, then $-\tau = \beta - \alpha$. This means, using the relation (3.62) that the matrix V must be hermitian, in any dimension:

$$V_{\alpha,\beta} = \bar{V}_{\beta,\alpha} . \quad (3.63)$$

In two dimensions if we consider a given τ of length between 0 and $2E$, there are typically roughly two pairs of momenta of length E , forming a rhombus such that $\tau = \alpha - \beta$. These two pairs are (α, β) and $(-\beta, -\alpha)$. Therefore the matrix V in addition to being hermitian has an additional symmetry

$$V_{\alpha,\beta} = V_{-\beta,-\alpha} \quad (3.64)$$

called the flip symmetry in [1]. Under this symmetry, two coefficients in e.g. the upper triangle of the matrix correspond to the same random variable. This means that V is not exactly a GUE. We can label the sectors of the circle in such a way that these two identified coefficients are symmetric with respect to the antidiagonal of the matrix. For that it suffices to divide the circle into an even number $2N$ of sectors labeled as $\{1, \dots, N\} \cup \{-N, \dots, -1\}$, so that $\{1, \dots, N\}$ labels projective sectors, as shown in Figure 4. Of course this is only a generic situation, and when the rhombus degenerates the answers are different. For instance for a momentum τ of length 2, there is only one pair $(\alpha, \beta) = (\tau/2, -\tau/2)$ such that $\tau = \alpha - \beta$, and for a momentum τ of length 0, every pair (α, α) is a solution. This means that all coefficients on the diagonal of the matrix V are equal. Intermediate situations when the rhombus almost degenerates

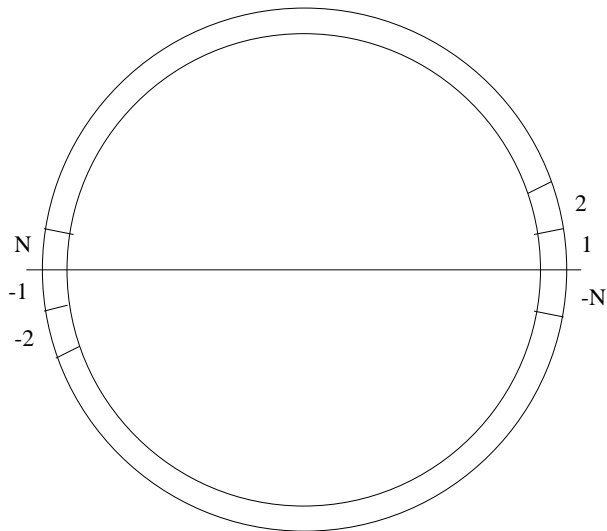


Figure 4: Numbering sectors

should also complicate the flip symmetry, and these issues are discussed in some detail below.

In three dimensions for a given τ there is typically an orbit of \sqrt{N} pairs of sectors (α, β) such that $\tau = \alpha - \beta$, forming a cone. The discrete Z_2 flip symmetry is therefore replaced by a $U(1)$ symmetry; the set of upper diagonal coefficients of the matrix V are therefore split into $O(N^{3/2})$ orbits under this symmetry, each orbit containing roughly \sqrt{N} coefficients. Clearly this matrix model is farther from the GUE model, but still the density of states should fall in the same category.

Having recognized the analogy, in phase space, between the Anderson model and random matrices, we shall now review first the regime $|p^2 - E| \gg \lambda^2 \pi^2$, then the last shell $|p^2 - 1| \simeq \lambda^2 \pi^2$.

3.3 The regime $|p^2 - 1| \gg \lambda^2 \pi^2$

This is the region in phase space where by definition the deterministic part $-\Delta - 1$ is forced to be not too small, and where the random part λV is statistically very small compared to this deterministic part. However the distribution for V is not compactly supported. Therefore the probability for V to have a norm comparable in size to $-\Delta - 1$ is never zero, even in that regime. However it is so small that if such an event happens in a phase space region R , there is such a small associated probability factor that it can pay for a very crude bound, obtained by translating directly the integration contour for the potential V in that region R . In this way we can prove not only that such an event has small probability, but that it does not influence other regions of phase space.

The corresponding rigorous analysis has been performed in great detail in the work of G. Poirot [29], and here we give only a rough sketch of the argument.

To get an idea of the respective sizes of the deterministic and random pieces in $(H - 1)$ we can first work at fixed j for a shell-conserving piece $V_{s,j}$. Consider first the deterministic part $p^2 - 1$. Restricted to the j -th shell it is by definition of that shell an operator with a norm of order M^{-j} . We want to prove that when restricted to that shell, the random part has typically a much smaller norm. If this is the case ordinary perturbation theory of the random part should be trusted.

The full interacting averaged Green's function is

$$\frac{\kappa(p)}{p^2 - 1 - E + i\varepsilon} (1 + \lambda V \frac{\kappa(p)}{p^2 - 1 - E + i\varepsilon})^{-1} . \quad (3.65)$$

We have divided into shells through $\kappa = \sum_j \kappa_j$. For a slightly more symmetric form we can put square roots of the free propagator on each side of V , and write $\kappa_j = \eta_j^2$. Then the random operator restricted to the j -th shell is

$$H_j(p, q) = \frac{\eta_j(p)}{(p^2 - 1 - E + i\varepsilon)^{1/2}} \lambda V(p - q) \frac{\eta_j(q)}{(q^2 - 1 - E + i\varepsilon)^{1/2}} \quad (3.66)$$

with

$$V(p - q) = \int dx e^{ix \cdot (p - q)} V(x) . \quad (3.67)$$

The main problem to define and control the averaged Green's function is now to compare the norm of H_j to 1 in order to invert the operator $1 + H_j$.

The sectors in a cube are roughly L^2 normalized characteristic functions $e_\alpha = M^{dj/2} \chi_\alpha(p)$, where the factor $M^{dj/2}$ is required for normalization.

The operator H_j restricted to a spatial cube of size M^j is now analogous to a matrix between sectors with coefficients

$$\begin{aligned} H_{\alpha,\beta} &\simeq \langle e_\alpha, H e_\beta \rangle \\ &= M^{dj} \int dp dq \frac{\chi_\alpha(p)}{(p^2 - 1 - E + i\varepsilon)^{1/2}} \lambda V(p - q) \frac{\chi_\beta(q)}{(q^2 - 1 - E + i\varepsilon)^{1/2}} \end{aligned} \quad (3.68)$$

since $\eta_j \chi_\alpha \simeq \chi_\alpha$, $\eta_j \chi_\beta \simeq \chi_\beta$. In a shell of index j , we have roughly $(p^2 - 1 + i\varepsilon)^{-1/2} = M^{j/2}$, so that the covariance between coefficients can be computed as

$$\begin{aligned} \langle H_{\alpha,\beta}, \bar{H}_{\gamma,\delta} \rangle_V &\simeq M^{(2d+2)j} \lambda^2 \int dp dq dp' dq' \chi_\alpha(p) \chi_\beta(q) \chi_\gamma(p') \chi_\delta(q') \\ &\int dx dy e^{ix \cdot (p - q)} e^{-iy \cdot (p' - q')} \langle V(x) V(y) \rangle_V . \end{aligned} \quad (3.69)$$

Using $\langle V(x)V(y) \rangle_V = \delta(x - y)$ we find

$$\begin{aligned}
\langle H_{\alpha,\beta}, H_{\gamma,\delta} \rangle_V &\simeq M^{(2d+2)j} \lambda^2 \int dp dq dp' dq' \\
&\quad \chi_\alpha(p) \chi_\beta(q) \chi_\gamma(p') \chi_\delta(q') \delta((p - q) - (p' - q')) \\
&\simeq M^{(2d+2)j} \lambda^2 \int dp dq dp' \chi_\alpha(p) \chi_\beta(q) \chi_\gamma(p') \chi_\delta(p' - p + q) \\
&\simeq \lambda^2 (\delta_{\alpha,\gamma} \delta_{\beta,\delta} + \delta_{\alpha,-\delta} \delta_{\beta,-\gamma}) \quad \text{for } d = 2. \tag{3.70}
\end{aligned}$$

In the last line we specialize to the generic momentum conservation rules for $d = 2$ (non-degenerate rhombuses)².

The random operator H_j in two dimensions restricted to a single cube, is therefore similar to a random matrix of the GUE type with the additional flip or $U(1)$ symmetry, and with covariance proportional to λ^2 . The size of the matrix is $N \sim M^j$, the number of isotropic sectors. We know the average norm of a N by N GUE matrix to be not much more than $O(1)$ for matrices with covariance $1/N$ on the independent coefficients (see Lemma 1 in section 2). Rescaling to covariance λ^2 is like rescaling H by $\lambda\sqrt{N} \sim \lambda M^{j/2}$. Hence we conclude that statistically the norm of the random part H_j should not be much larger than $O(1)\lambda M^{j/2}$. This means it is statistically much smaller than 1 for $\lambda M^{j/2} \ll 1$, which is nothing but $j \leq j_{\max}(\lambda)$ (if M is large). We conclude that statistically we can perturb the random part with respect to the deterministic part in the whole regime $|p^2 - 1| \gg \lambda^2$, as announced.

Of course this is a very rough argument and we must provide details on what to do in the infinite volume case, and how to treat the statistically rare events where the random part is unfortunately much larger than expected.

Since the space volume can be large, one performs a battery of tests which for each cube Δ tell us whether the random part in that cube Δ has an exceptionally large norm or not. This battery of tests is called a large/small field analysis in constructive theory.

Roughly speaking one uses then an analyticity argument in the cubes where V has anomalously large norm, which form the so-called large field region. The contour of integration for the V 's of this large field regions is translated to create an imaginary part of same order than the deterministic part, and a bound is applied to a resolvent expansion which tests whether the Green's function effectively or not has visited these large field regions [29].

Taking absolute values for this bound destroys oscillations so we need to pay for a large factor corresponding to the apparent change in the Gaussian normalization:

$$|e^{-(V+ia)^2/c}|/e^{-V^2/c} = e^{+a^2/c} \tag{3.71}$$

but this large factor is compensated by the small probabilistic factor coming from the large field condition. To understand roughly why this is possible in

²For general d we would find a factor $\lambda^2 M^{(2-d)j}$ times the constraint that there are momenta p, q and p' in the sectors α, β, γ such that $p' - p + q$ is in δ , and this is a more complicated constraint.

two dimensions, where $N \simeq M^j$, recall that to create with λV an imaginary part of size M^{-j} requires a translation of the zero momentum V_0 coefficient on the diagonal of the matrix V by $i\lambda^{-1}M^{-j}$. Then the factor to pay following (3.71) is $e^{a^2/c}$ with $a = \lambda^{-1}M^{-j}$ and the covariance c for the zero momentum coefficient is M^{-2j} . So the factor to pay is $e^{\lambda^{-2}}$. But if the norm of V is larger than $\lambda^{-1}M^{-j}$, by lemma 1, the small probabilistic factor is of order $e^{-M^{j/3}\lambda^{-2}/3}$ so it compensates for much more than the factor to pay. This is what essentially fuels the analysis of [29].

This large field analysis is also complicated by some auxiliary expansions to take into account the non-diagonal shell changing parts of the potential, which does not change the overall picture. Altogether this analysis, although technically tedious, is obviously much cruder than a shift of the expansion point, which becomes necessary to treat the second regime.

The conclusion is that in this regime one can invert the operator $H - E$, and the averaged Green's function restricted to that regime has the expected size and spatial scaled decay rate in $|xM^{-j_{max}}| = \lambda^2|x|$.

4 The regime $|p^2 - 1| \leq c\lambda^2$

We know heuristically from first order perturbation theory that the retarded or advanced Green's functions should decay with a rate of order $\lambda^2\pi^2$. Therefore it makes sense, at the level of a single Green's function or for the density of states to treat this regime as a single momentum slice. Expecting this decay, it is enough to understand the model in a single space cube of side size λ^{-2} , and then to treat the full model through some cluster expansion. In this section we want to summarize the more detailed computations in [1] which prove that the density of state of a single matrix with the \mathbb{Z}_2 flip symmetry has the same large N limit than a GUE matrix. The core of the argument reproduces the supersymmetric computation of section 2.3, but with some additional auxiliary fields to take into account the flip symmetry. It is then shown that the presence of these fields do not modify the main saddle point and the leading contribution to the density of states.

4.1 The Flip Matrix model

The main idea in [1] is to introduce a discrete version of this single cube problem, in which phase space and the 2d momentum conservation rules have been discretized and somewhat simplified, according to the isotropic discretization and to equations (3.63-3.64) of subsection 3.2.

Using the labeling of Fig 1, the structure of V can be summarized as follows (remember rows and columns are labeled as $\{1, \dots, N, -N, \dots, -1\}$ hence *not*

as $\{1, \dots, N, -1, \dots, -N\}$):

$$V = \left(\begin{array}{c|c} \left[\begin{array}{cccc} v_0 & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & v_0 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] & \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & v_{\alpha, -\beta} & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] \\ \hline \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & \overline{v_{\alpha, \beta}} & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] & \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & v_{\beta, -\beta} & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] \\ \hline \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & \overline{v_{\alpha, -\beta}} & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] & \left[\begin{array}{cccc} v_0 & \vdots & \vdots & \\ \vdots & \ddots & \vdots & \\ \vdots & \vdots & v_0 & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right] \end{array} \right)$$

and the main result of [1] is:

Theorem 1 *In the large N -limit, the DOS of this flip matrix model converges to Wigner's semi-circle distribution. The corrections to the limit are uniformly bounded as $O(1/N)$ as $N \rightarrow \infty$.*

Let us summarize now the proof of this theorem.

4.2 The isotropic 2d flip model: Fermionic terms

Introducing supersymmetric fields

$$\Psi_{\sigma\alpha} = (S_{\sigma\alpha}, \chi_{\sigma\alpha}),$$

for $\sigma = \pm 1$ and Greek variables such as α, β now running only within $[1, \dots, N]$. we find the following formula for the density of states

$$\nu(E) = \lim_{\Im E \rightarrow 0^+} \frac{-1}{\pi} \Im \int \frac{i}{2N} \sum_{\alpha, \sigma} S_{\alpha\sigma}^+ S_{\alpha\sigma} e^{iL_0} \prod_{\alpha, \sigma} d\Psi_{\sigma\alpha}^+ d\Psi_{\sigma\alpha} d\mu(V), \quad (4.72)$$

where the supersymmetric action decomposes as

$$L_0 = (A_0 + B_0 + C_0),$$

with

$$\begin{aligned}
A_0 &= \sum_{\alpha < \beta, \sigma} \{ \lambda V_{\alpha, \sigma \beta} (\Psi_{\alpha}^+ \Psi_{\sigma \beta} + \Psi_{-\sigma \beta}^+ \Psi_{-\alpha}) + h.c. \}, \\
B_0 &= \sum_{\alpha} \lambda V_{\alpha, -\alpha} (\Psi_{\alpha}^+ \Psi_{-\alpha} + h.c.), \\
C_0 &= (E + \lambda V_0) \sum_{\alpha, \sigma} \Psi_{\sigma \alpha}^+ \Psi_{\sigma \alpha}.
\end{aligned}$$

The Gaussian integration over the V variables will be performed except for V_0 , leading to the following quartic action

$$\nu(E) = \lim_{Im E \rightarrow 0^+} \frac{1}{\pi} Im \int S_{\alpha}^+ S_{\alpha} e^{-L} \prod_{\alpha, \sigma} d\Psi_{\sigma \alpha}^+ d\Psi_{\sigma \alpha} d\mu_0(V_0),$$

$$\begin{aligned}
L &= \lambda^2 (A + B + C), \\
A &= \sum_{\alpha < \beta, \sigma} (\Psi_{\alpha}^+ \Psi_{\sigma \beta} + \Psi_{-\sigma \beta}^+ \Psi_{-\alpha}) (\Psi_{\sigma \beta}^+ \Psi_{\alpha} + \Psi_{-\alpha}^+ \Psi_{-\sigma \beta}), \\
B &= \sum_{\alpha} \Psi_{\alpha}^+ \Psi_{-\alpha} \Psi_{-\alpha}^+ \Psi_{\alpha}, \\
C &= (-i\lambda^{-2} E - i\lambda^{-1} V_0) \sum_{\alpha, \sigma} \Psi_{\sigma \alpha}^+ \Psi_{\sigma \alpha}.
\end{aligned} \tag{4.73}$$

Some tedious rewriting using the commutation and anticommutation rules eventually leads to $L = \lambda^2 (\mathcal{A} + \mathcal{B} + \mathcal{C})$ with

$$\begin{aligned}
\mathcal{A} &= (1/2) \sum_{\alpha, \beta, \sigma} (\Psi_{\alpha}^+ \Psi_{\sigma \beta} + \Psi_{-\sigma \beta}^+ \Psi_{-\alpha}) (\Psi_{\sigma \beta}^+ \Psi_{\alpha} + \Psi_{-\alpha}^+ \Psi_{-\sigma \beta}), \\
\mathcal{B} &= \sum_{\alpha} \mathcal{B}_{\alpha}, \\
\mathcal{B}_{\alpha} &= -2S_{\alpha}^+ S_{-\alpha} S_{-\alpha}^+ S_{\alpha} - \frac{1}{2} \sum_{\sigma} (\Psi_{\sigma \alpha}^+ \Psi_{\sigma \alpha})^2 \\
&\quad - \left(\sum_{\sigma} S_{\sigma \alpha}^+ \chi_{-\sigma \alpha}^+ \right) \left(\sum_{\sigma} S_{\sigma \alpha} \chi_{-\sigma \alpha} \right), \\
\mathcal{C} &= C = (-i\lambda^{-2} E - i\lambda^{-1} V_0) \sum_{\alpha, \sigma} \Psi_{\sigma \alpha}^+ \Psi_{\sigma \alpha}.
\end{aligned} \tag{4.74}$$

The term \mathcal{B} being a sum of *diagonal quartic* terms, will be neglected for a while. For indeed it cannot be written as the *square* of a sum over α . However, the bosonic part of this term has the *wrong sign*, which may create some difficulties when performing the integration in (4.72). This problem will be addressed later.

The other terms can be reorganized so as to get the square of a sum over the α 's by pushing the terms with index α on the left and the ones with index β

on the right. This must be done with care according to the commutation rules for Bosons and for fermions. The calculation is tedious but straightforward and gives:

$$\begin{aligned}
\mathcal{A} = & (1/2) \left(\sum_{\alpha, \sigma} S_{\sigma\alpha}^+ S_{\sigma\alpha} \right)^2 - (1/2) \left(\sum_{\alpha, \sigma} \chi_{\sigma\alpha}^+ \chi_{\sigma\alpha} \right) (h.c.) \\
& + 2 \left(\sum_{\alpha} S_{\alpha}^+ S_{-\alpha}^+ \right) (h.c.) + \left(\sum_{\sigma\alpha} S_{\sigma\alpha} \chi_{\sigma\alpha}^+ \right) (h.c.) \\
& + \left(\sum_{\sigma\alpha} S_{\sigma\alpha}^+ \chi_{-\sigma\alpha}^+ \right) (h.c.). \tag{4.75}
\end{aligned}$$

The squares can be *unfolded* by mean of an integration over auxiliary gaussian fields. This amount to introduce two real fields a_0 and b_0 , one complex field a, \bar{a} and two pairs of Fermionic fields $\xi, \bar{\xi}$ and $\eta, \bar{\eta}$ with Gaussian measure

$$\begin{aligned}
d\mu(a_0, b_0, a, \bar{a}, \bar{\xi}, \xi, \bar{\eta}, \eta) = & e^{-V_0^2/2 - a_0^2/2 - b_0^2/2 - |a|^2/2 - \xi^* \xi - \eta^* \eta} \\
& \times \dots \times \frac{dV_0 da_0 db_0 d^2 a}{2(2\pi)^{1/2}} d\xi^* d\xi d\eta^* d\eta. \tag{4.76}
\end{aligned}$$

Therefore

$$e^{-\lambda^2(A+C)} = \int d\mu \exp \left(i\lambda \sum_{\alpha} \Phi_{\alpha}^+ R \Phi_{\alpha} \right),$$

with

$$\begin{aligned}
\sum_{\alpha} \Phi_{\alpha}^+ R \Phi_{\alpha} = & (a_0 + V_0 + \lambda^{-1} E) \sum_{\alpha, \sigma} S_{\sigma\alpha}^+ S_{\sigma\alpha} \\
& + (ib_0 + V_0 + \lambda^{-1} E) \sum_{\alpha, \sigma} \chi_{\sigma\alpha}^+ \chi_{\sigma\alpha} + \left(\bar{a} \sum_{\alpha} S_{\alpha} S_{-\alpha} + h.c. \right) \\
& + \left(\xi^* \sum_{\sigma\alpha} S_{\sigma\alpha}^+ \chi_{\sigma\alpha} + h.c. \right) + \left(\eta^* \sum_{\sigma\alpha} S_{\sigma\alpha} \chi_{-\sigma\alpha} + h.c. \right),
\end{aligned}$$

here Φ is the superfield $\Phi_{\alpha}^+ = (S_{\alpha}^+, S_{-\alpha}, \chi_{\alpha}^+, \chi_{-\alpha})$ and R is a 4×4 supermatrix. Setting $A_0 = (a_0 + V_0 + \lambda^{-1} E)$ and $\imath B_0 = (\imath b_0 + V_0 + \lambda^{-1} E)$ we obtain the following representation

$$\nu(E) = \frac{-1}{\pi} \Im \int d\mu [\lambda R]_{bb,11}^{-1} [Sdet(\imath \lambda R)]^{-N}. \tag{4.77}$$

The matrix R is a so-called 4x4 super-matrix

$$R = \begin{pmatrix} A_0 & a & \xi^* & -\imath\eta \\ \bar{a} & A_0 & \eta^* & -\imath\xi \\ \xi & \eta & \imath B_0 & 0 \\ -\imath\eta^* & -\imath\xi^* & 0 & \imath B_0 \end{pmatrix} = \begin{pmatrix} A & \rho^* \\ \rho & B \end{pmatrix}. \tag{4.78}$$

and the coefficient $[\lambda R]_{bb,11}^{-1}$ is a particular (bosonic) coefficient of the inverse matrix, the one corresponding to first row and column. In this computation

we use the fact that this coefficient is also equal to $[\lambda R]_{bb,22}^{-1}$ and that we get therefore $2N$ equal terms from the sum over α and σ in (4.72), and cancel this sum with the $1/2N$ normalization factor.

We perform a rescaling of all the variables by $\lambda^{-1} = \sqrt{2N}$ and write

$$\begin{aligned} \nu(E) &= \frac{-1}{\pi} \Im \frac{\sqrt{N}}{2\sqrt{\pi}} \int da_0 db_0 da d\bar{a} dV_0 d\xi^* d\xi d\eta^* d\eta \\ &\quad e^{-N(a_0^2 + b_0^2 + |a|^2 + V_0^2 + 2(\xi^* \xi + \eta^* \eta))} \\ &\quad \left[(A_0 - \frac{1}{iB_0}(\xi^* \xi + \eta^* \eta))^2 - (a - \frac{2}{iB_0} \xi^* \eta)(\bar{a} - \frac{2}{iB_0} \eta^* \xi) \right]^{-N-1} \\ &\quad [A_0 - \frac{1}{iB_0}(\xi^* \xi + \eta^* \eta)](iB_0)^{2N}, \end{aligned} \quad (4.79)$$

defining $A_0 = a_0 + V_0 + E$, $iB_0 = ib_0 + V_0 + E$.

We want to compute the fermionic integration

$$\begin{aligned} I_f &= \int d\xi^* d\xi d\eta^* d\eta e^{-2N(\xi^* \xi + \eta^* \eta)} \\ &\quad \frac{[1 - \frac{1}{iB_0 A_0}(\xi^* \xi + \eta^* \eta)]}{\left[(A_0 - \frac{1}{iB_0}(\xi^* \xi + \eta^* \eta))^2 - (a - \frac{2}{iB_0} \xi^* \eta)(\bar{a} - \frac{2}{iB_0} \eta^* \xi) \right]^{N+1}}. \end{aligned} \quad (4.80)$$

Note that

$$e^{-2N(\xi^* \xi + \eta^* \eta)} = 1 - 2N(\xi^* \xi + \eta^* \eta) + 4N^2 \xi^* \xi \eta^* \eta \quad (4.81)$$

$$\begin{aligned} &\left[(A_0 - \frac{1}{iB_0}(\xi^* \xi + \eta^* \eta))^2 - (a - \frac{2}{iB_0} \xi^* \eta)(\bar{a} - \frac{2}{iB_0} \eta^* \xi) \right]^{-N-1} \\ &= \left[(A_0^2 - |a|^2) - 2\frac{A_0}{iB_0}(\xi^* \xi + \eta^* \eta) \right. \\ &\quad \left. + \frac{2}{iB_0}(a\eta^* \xi + \bar{a}\xi^* \eta) + \frac{6}{(iB_0)^2} \xi^* \xi \eta^* \eta \right]^{-N-1} \\ &= \frac{1}{(A_0^2 - |a|^2)^{N+1}} \left[1 - \frac{N+1}{A_0 - |a|^2} \left(-2\frac{A_0}{iB_0}(\xi^* \xi + \eta^* \eta) + \frac{2}{iB_0}(a\eta^* \xi + \bar{a}\xi^* \eta) \right. \right. \\ &\quad \left. \left. + \frac{6}{(iB_0)^2} \xi^* \xi \eta^* \eta \right) + \frac{(N+1)(N+2)}{A_0^2 - |a|^2} \frac{4}{(iB_0)^2} \xi^* \xi \eta^* \eta \right]. \end{aligned} \quad (4.82)$$

All terms vanish through fermionic integration except their terms in $\xi^* \xi \eta^* \eta$. So (4.80) can be computed as

$$\begin{aligned} I_f &= \frac{1}{4\pi^2(A_0^2 - |a|^2)^{N+1}} \left[\frac{N+1}{A_0^2 - |a|^2} \frac{4N+2}{(iB_0)^2} - \frac{8N(N+1)}{A_0^2 - |a|^2} \frac{A_0}{iB_0} + 4N^2 \right. \\ &\quad \left. - \frac{1}{iA_0 B_0} \left(\frac{N+1}{A_0^2 - |a|^2} \frac{4A_0}{iB_0} - 4N \right) \right] \\ &= \frac{1}{4\pi^2(A_0^2 - |a|^2)^{N+1}} \left[\frac{N+1}{A_0^2 - |a|^2} \left(\frac{4N-2}{(iB_0)^2} - \frac{8NA_0}{iB_0} \right) + 4N^2 + \frac{4N}{iA_0 B_0} \right]. \end{aligned} \quad (4.83)$$

We have therefore

$$\nu(E) = \frac{-1}{\pi} \Im \frac{\sqrt{N}}{2\sqrt{\pi}} \int da_0 db_0 dad\bar{a}dV_0 e^{-N(a_0^2+b_0^2+|a|^2+V_0^2)} (iB_0)^{2N} A_0 I_f, \quad (4.84)$$

hence

$$\begin{aligned} \nu(E) &= \frac{-\sqrt{N}}{4\pi^{7/2}} \Im \int d\mu_b (A_0^2 - |a|^2)^{-N-2} (iB_0)^{2N-2} \\ &\quad [(N+1)((2N-1)A_0 - 4iNA_0^2B_0) + 2iNB_0(A_0^2 - |a|^2)(1 + iNA_0B_0)] , \end{aligned} \quad (4.85)$$

where

$$d\mu_b = da_0 db_0 dad\bar{a}dV_0 e^{-N(a_0^2+b_0^2+|a|^2+V_0^2)}. \quad (4.86)$$

Expanding the Fermionic part and keeping only the leading terms in N^2 we would find

$$\begin{aligned} \nu(E) &= -\frac{N^{5/2}}{2\pi^{7/2}} \Im \int d\mu_b (A_0^2 - |a|^2)^{-N-2} (iB_0)^{2N-2} \\ &\quad A_0 [1 - 2iB_0A_0 + (iB_0)^2(A_0^2 - |a|^2) + O(1/N)] . \end{aligned} \quad (4.87)$$

We use the notations $\mathcal{E} = -E/2 + i\sqrt{1-E^2/4}$ so that $\mathcal{E}\mathcal{E}^* = 1$.

The ordinary saddle points correspond to $|a| = V_0 = 0$ and for the main saddle point $a_0 = ib_0 = \mathcal{E}$, so that $A_0 = iB_0 = -\mathcal{E}^*$ and $(iA_0B_0)^{-1} = (\mathcal{E})^2$.

At the second saddle point with same bosonic action $a_0 = \mathcal{E}, ib_0 = \mathcal{E}^*$ we have $iA_0B_0 = 1$ so that this contribution is $O(1/N)$.

Other saddle points have smaller actions [1].

The Hessian at the main saddle point is obtained by writing $a_0 \rightarrow \mathcal{E} + a_0$ and $ib_0 \rightarrow \mathcal{E} + ib_0$ (which includes a translation of the integration contour) and expanding to second order. The linear part vanishes at the saddle point and we find the Hessian approximation:

$$\begin{aligned} \nu(E) &= \frac{N^{5/2}}{2\pi^{7/2}} \Im \int da_0 db_0 dad\bar{a}dV_0 e^{-N[(\mathcal{E}+a_0)^2 + (-i\mathcal{E}+b_0)^2 + |a|^2 + V_0^2]} \\ &\quad e^{-N[\log[(\mathcal{E}+a_0+E+V_0)^2 - |a|^2] - 2\log(\mathcal{E}+ib_0+E+V_0)]} \\ &\quad \mathcal{E}[1 - 2\mathcal{E}^2 + \mathcal{E}^4] + O(1/N) \end{aligned} \quad (4.88)$$

$$\begin{aligned} &= \frac{N^{5/2}}{2\pi^{7/2}} \Im \int da_0 db_0 dad\bar{a}dV_0 \\ &\quad e^{-N[(a_0^2+b_0^2+|a|^2)(1-\mathcal{E}^2)+V_0^2+2(ib_0-a_0)V_0\mathcal{E}^2]} \end{aligned} \quad (4.89)$$

$$\mathcal{E}[1 - \mathcal{E}^2]^2 + O(1/N) . \quad (4.90)$$

We can compute the Gaussian integral over $|a|^2$ as $\frac{2\pi}{N(1-\mathcal{E}^2)}$. The remaining one is $(\pi/N)^{3/2}$ times $1/\sqrt{\det Q}$ where Q is the 3x3 matrix:

$$Q = \begin{pmatrix} 1 - \mathcal{E}^2 & 0 & -\mathcal{E}^2 \\ 0 & 1 - \mathcal{E}^2 & i\mathcal{E}^2 \\ -\mathcal{E}^2 & i\mathcal{E}^2 & 1 \end{pmatrix} . \quad (4.91)$$

Since $\det Q = (1 - \mathcal{E}^2)^2$ we find:

$$\nu(E) = \frac{1}{\pi} \Im \mathcal{E} = \frac{1}{\pi} \sqrt{1 - E^2/4} \quad (4.92)$$

which is the Wigner semi-circle law.

This essentially ends the proof, up to two main effects which have been neglected: we have to bound the error to the Hessian approximation to the saddle point and to treat the "diagonal" \mathcal{B} quartic terms.

4.3 Bounding the error to Hessian approximation

First of all, we have to check that the path we have chosen passing through the saddle is the correct one. This is true if the real part of the action has minimum at the saddle.

Then we extract the leading contribution from the observable term, (the corresponding integral is one by supersymmetry) that gives the semicircle law. The remaining error terms are of order $O(1/N)$. To prove that we need to divide the integration region in the five parameters in four main zones

- the vicinity of the ordinary saddle point: here the Hessian approximation is correct. As we have extracted the leading term already the numerator is small;
- the vicinity of the second saddle point: also here the Hessian approximation is correct, but we have to expand around the second saddle;
- a large but compact region around the two saddles: here we bound the integrand by supnorm;
- the rest: as we are very far from the saddle we can extract some exponential decay. This allows to perform the integral and gives an exponentially small factor.

Extracting the leading term

Putting together the terms of order N^2 , N and 1 we have

$$\nu(E) = \frac{-\sqrt{N}}{4\pi^{7/2}} \Im \int d\mu'_b \frac{A_0}{(A_0^2 - |a|^2)} [N^2 I_1 + N I_2 + I_3] \quad (4.93)$$

where the measure μ'_b contains also the logarithmic terms:

$$d\mu'_b = d\mu_b \left(\frac{(iB_0)^2}{(A_0^2 - |a|^2)} \right)^N, \quad (4.94)$$

and

$$\begin{aligned}
I_1 &= \frac{1}{(iB_0)^2(A_0^2 - |a|^2)} 2 [1 - 2iB_0A_0 + (iB_0)^2(A_0^2 - |a|^2)] , \\
I_2 &= \frac{1}{(iB_0)^2(A_0^2 - |a|^2)} \left[1 - 4A_0iB_0 + 2\frac{iB_0}{A_0}(A_0^2 - |a|^2) \right] , \\
I_3 &= -\frac{1}{(iB_0)^2(A_0^2 - |a|^2)} .
\end{aligned} \tag{4.95}$$

Note that, if we have no observable, the full supersymmetric integral is 1, and its approximation to leading order is also 1! So we must have

$$1 = N^2 \frac{\sqrt{N}}{4\pi^{5/2}} \int d\mu'_b I_1 . \tag{4.96}$$

Therefore

$$\nu(E) = +\frac{1}{\pi} \Im \mathcal{E} + N^2 R_1 + N R_2 + R_3 , \tag{4.97}$$

where

$$\begin{aligned}
R_1 &= \frac{-\sqrt{N}}{4\pi^{7/2}} \Im \int d\mu'_b \left[\frac{A_0}{(A_0^2 - |a|^2)} + \mathcal{E} \right] I_1 , \\
R_2 &= \frac{-\sqrt{N}}{4\pi^{7/2}} \Im \int d\mu'_b \frac{A_0}{(A_0^2 - |a|^2)} I_2 , \\
R_3 &= \frac{-\sqrt{N}}{4\pi^{7/2}} \Im \int d\mu'_b \frac{A_0}{(A_0^2 - |a|^2)} I_3 .
\end{aligned} \tag{4.98}$$

We need to prove that the three quantities $N^2|R_1|$, $N|R_2|$ and $|R_3|$ are $O(1/N)$.

Translation to the saddle

We translate to the saddle

$$a_0 \rightarrow a_0 + \mathcal{E}, \quad b_0 \rightarrow b_0 - i\mathcal{E} . \tag{4.99}$$

We do not cross any singularity (thanks to the $i\varepsilon$ regulator) hence the integral does not change. The measure becomes

$$d\mu'_b = da_0 db_0 da d\bar{a} dV_0 e^{-NS(a_0, b_0, V_0, |a|)} \tag{4.100}$$

and

$$\begin{aligned}
S &= [(\mathcal{E} + a_0)^2 + (-i\mathcal{E} + b_0)^2 + |a|^2 + V_0^2] \\
&\quad + [\log[(-\mathcal{E}^* + a_0 + V_0)^2 - |a|^2] - 2\log(-\mathcal{E}^* + ib_0 + V_0)] \\
&= [a_0^2 + b_0^2 + 2\mathcal{E}(a_0 - ib_0) + |a|^2 + V_0^2] \\
&\quad + [\log[(-\mathcal{E}^* + a_0 + V_0)^2 - |a|^2] - 2\log(-\mathcal{E}^* + ib_0 + V_0)] .
\end{aligned} \tag{4.101}$$

In order to bound the error terms, we will need a bound on the absolute value of the measure. So we have to study the critical points of $\text{Re } S$:

$$\text{Re } S = a_0^2 + b_0^2 + 2b_0\mathcal{E}_i - 2a_0\mathcal{E}_r + |a|^2 + V_0^2 + \frac{1}{2} \ln F_a - \ln F_b \quad (4.102)$$

where we defined $\mathcal{E}_i = \sqrt{1 - E^2/4}$, $\mathcal{E}_r = E/2$,

$$\begin{aligned} F_a &= x^2[x^2 + 2(\mathcal{E}_i^2 - |a|^2)] + (\mathcal{E}_i^2 + |a|^2)^2 \\ F_b &= (b_0 + \mathcal{E}_i)^2 + (V_0 + \mathcal{E}_r)^2 \end{aligned} \quad (4.103)$$

and

$$x = a_0 + V_0 + \mathcal{E}_r . \quad (4.104)$$

The equations for the critical points are

$$\begin{aligned} 0 &= \partial_{a_0} \text{Re } S = 2 \left\{ a_0 - \mathcal{E}_r + \frac{x}{F_a} [x^2 + \mathcal{E}_i^2 - |a|^2] \right\} , \\ 0 &= \partial_{b_0} \text{Re } S = 2(b_0 + \mathcal{E}_i) \left[1 - \frac{1}{F_b} \right] , \\ 0 &= \partial_a \text{Re } S = 2\bar{a} \left[1 + \frac{1}{F_a} (-x^2 + \mathcal{E}_i^2 + |a|^2) \right] , \\ 0 &= \partial_{V_0} \text{Re } S = 2 \left\{ V_0 \left(1 - \frac{1}{F_b} \right) - \frac{1}{F_b} \mathcal{E}_r + \frac{x}{F_a} [x^2 + \mathcal{E}_i^2 - |a|^2] \right\} . \end{aligned} \quad (4.105)$$

We remark that $E = O(\lambda^2) = O(1/N)$. Therefore we can take $E = 0$ in the saddle point computation, as it only gives corrections of order $O(1/N)$. At $E = 0$ we have $\mathcal{E}_r = 0$ and $\mathcal{E}_i = 1$. The saddle equations become

$$\begin{aligned} 0 &= \partial_{a_0} \text{Re } S = 2 \left\{ a_0 + \frac{x}{F_a} (x^2 + 1 - |a|^2) \right\} , \\ 0 &= \partial_{b_0} \text{Re } S = 2(b_0 + 1) \left[1 - \frac{1}{F_b} \right] , \\ 0 &= \partial_a \text{Re } S = 2\bar{a} \left[1 + \frac{1}{F_a} (-x^2 + 1 + |a|^2) \right] , \\ 0 &= \partial_{V_0} \text{Re } S = 2 \left\{ V_0 \left(1 - \frac{1}{F_b} \right) + \frac{x}{F_a} (x^2 + 1 - |a|^2) \right\} , \end{aligned} \quad (4.106)$$

where now $x = a_0 + V_0$,

$$\begin{aligned} F_a &= x^2[x^2 + 2(1 - |a|^2)] + (1 + |a|^2)^2 , \\ F_b &= (b_0 + 1)^2 + V_0^2 . \end{aligned} \quad (4.107)$$

Note that, from the first and last equation we have

$$V_0 \left(1 - \frac{1}{F_b} \right) = -\frac{x}{F_a} (x^2 + 1 - |a|^2) = a_0 . \quad (4.108)$$

Moreover, from the third equation we have $a = 0$ or $(1 - x^2 + |a|^2) = -F_a$. In this last case we have to solve

$$x^2[x^2 + 2(1 - |a|^2)] + (1 + |a|^2)^2 = -(1 - x^2 + |a|^2) . \quad (4.109)$$

We find there is no real solution for $y = x^2$. Therefore we have always $a = 0$ in the following and $F_a = (1 + x^2)^2$. The saddle equations then reduce to

$$\begin{aligned} 0 &= a_0 + \frac{x}{x^2 + 1} , \\ 0 &= (b_0 + 1) \left[1 - \frac{1}{F_b} \right] , \\ 0 &= V_0 \left(1 - \frac{1}{F_b} \right) + \frac{x}{1 + x^2} . \end{aligned} \quad (4.110)$$

Now we distinguish several cases. From the second equation we see that $b_0 + 1 = 0$ or $F_b = 1$.

First case: $F_b = 1$

From (4.108) we have $a_0 = 0$. We also have $x = V_0$, and

$$\frac{x}{1 + x^2} = \frac{V_0}{1 + V_0^2} = 0 , \quad (4.111)$$

which implies $V_0 = 0$. Finally b_0 is given by

$$1 = F_b = (b_0 + 1)^2 \quad \text{which implies} \quad b_0 = 0, b_0 = -2 . \quad (4.112)$$

Therefore we have two saddle points: $a_0 = V_0 = a = 0$ and $b_0 = 0, -2$.

Second case: $b_0 + 1 = 0$

In this case we have $F_b = V_0^2$, $a_0 = V_0 - 1/V_0$ and $x = (2V_0^2 - 1)/V_0$. Inserting the expressions in terms of V_0 in the first equation we find

$$\frac{V_0^2 - 1}{V_0} + \frac{V_0(2V_0^2 - 1)}{4V_0^4 - 3V_0^2 + 1} = 0 \quad (4.113)$$

which gives the equation

$$4z^3 - 5z^2 + 3z - 1 = 0 \quad (4.114)$$

where $z = V_0^2$. Studying the first derivative of this expression we see that there is only one real solution. This solution is positive with $1/2 < z_s < 1$. So there are two solutions $V_0 = \pm\sqrt{z_s}$. We have then two new critical points in $a = 0$, $b_0 = -1$, $V_0 = \pm\sqrt{z_s}$ and $a_0 = (V_0^2 - 1)/V_0$.

So finally we have identified four critical points:

$$\begin{aligned} S_1 &\equiv a_0 = V_0 = a = b_0 = 0 , \\ S_2 &\equiv a_0 = V_0 = a = 0, \quad b_0 = -2 , \\ S_3 &\equiv a = 0, \quad b_0 = -1, \quad a_0 = \frac{(z_s - 1)}{\sqrt{z_s}}, \quad V_0 = \sqrt{z_s} , \\ S_4 &\equiv a = 0, \quad b_0 = -1, \quad a_0 = -\frac{(z_s - 1)}{\sqrt{z_s}}, \quad V_0 = -\sqrt{z_s} . \end{aligned} \quad (4.115)$$

The real part of the action at the leading order ($E = 0$) is

$$\text{Re}_{E=0}S = a_0^2 + b_0^2 + 2b_0 + |a|^2 + V_0^2 + \frac{1}{2} \ln F_a - \ln F_b . \quad (4.116)$$

Is is easy to see that in S_1 and S_2 we have $\text{Re}_{E=0}S = 0$. On the other hand in S_3, S_4 the action is (writing z for z_s):

$$\text{Re}_{E=0}S = \frac{(2z-1)(z-1)}{z} + \log \frac{4z^2 - z + 1}{z^2} \quad (4.117)$$

which since $1/2 < z < 1$ satisfies

$$\text{Re}_{E=0}S > -1/4 + \log(3/2) > 0.15 . \quad (4.118)$$

Integration regions

We call $X = (a_0, b_0, V_0, a)$ so that the integral to study is

$$\int dX e^{-N\text{Re}S(X)} |R(X)| \quad (4.119)$$

where $R(X)$ depends on which error term we are looking at. We introduce the norm

$$\|X\|^2 = X^*X = a_0^2 + b_0^2 + |a|^2 + V_0^2 . \quad (4.120)$$

Now we partition the integration domain in four regions T_i , $i = 1, \dots, 4$, where

$$\begin{aligned} T_1 &= \left\{ X \mid \|X\| \leq \frac{1}{N^{1/3+\delta}} \right\} , \\ T_2 &= \left\{ X \mid \|X - S_2\| \leq \frac{1}{N^{1/3+\delta}} \right\} , \\ T_3 &= \{ X \mid \|X\| < K \text{ and } X \notin T_1 \cup T_2 \} , \\ T_4 &= \{ X \mid \|X\| > K \} , \end{aligned} \quad (4.121)$$

where $0 < \delta \ll 1$ and $K \gg 1$ are fixed constants.

Bound in the region T_1

In this region we can expand the action to the second order. We get

$$\begin{aligned} N\text{Re}S &= N[(a_0^2 + b_0^2 + |a|^2)2\mathcal{E}_i^2 + V_0^2 + 2[b_0 2\mathcal{E}_r \mathcal{E}_i + a_0(\mathcal{E}_i^2 - \mathcal{E}_r^2)]V_0 \\ &\quad + O(\|X\|^3)] \\ &= NX^*HX + O(N^{-3\delta}) \end{aligned} \quad (4.122)$$

where

$$H = \begin{pmatrix} 2\mathcal{E}_i^2 & 0 & \mathcal{E}_i^2 - \mathcal{E}_r^2 & 0 \\ 0 & 2\mathcal{E}_i^2 & 2\mathcal{E}_r \mathcal{E}_i & 0 \\ \mathcal{E}_i^2 - \mathcal{E}_r^2 & 2\mathcal{E}_r \mathcal{E}_i & 1 & 0 \\ 0 & 0 & 0 & 2\mathcal{E}_i^2 \end{pmatrix} = \begin{pmatrix} \text{Re}Q & 0 \\ 0 & 2\mathcal{E}_i^2 \end{pmatrix} \quad (4.123)$$

where Q was defined in (4.91).

Note that H is positive definite, so the measure is well defined. Now we remark that in this region

$$\begin{aligned} I_1 &= (1 - \mathcal{E}^2)^2 + O(\|X\|) , \\ I_2 &= \mathcal{E}^2(\mathcal{E}^2 - 2) + O(\|X\|) , \\ I_3 &= -\mathcal{E}^4 + O(\|X\|) , \end{aligned} \quad (4.124)$$

and

$$\left[\frac{A_0}{(A_0^2 - |a|^2)} + \mathcal{E} \right] = O(\|X\|) . \quad (4.125)$$

Inserting this bounds in R_1, R_2, R_3 we get

$$\begin{aligned} N^2 |R_1|_{T_1} &\leq \text{const } N^{5/2} \int dX e^{-NX^* H X} O(\|X\|) = O\left(\frac{1}{\sqrt{N}}\right) , \\ N |R_2|_{T_1} &\leq \text{const } N^{3/2} \int dX e^{-NX^* H X} = O\left(\frac{1}{N}\right) , \\ |R_3|_{T_1} &\leq \text{const } \sqrt{N} \int dX e^{-NX^* H X} = O\left(\frac{1}{N^2}\right) . \end{aligned} \quad (4.126)$$

By parity we can improve the first estimate and obtain in fact rather easily $N^2 R_1 = O(1/N)$.

Bound in the region T_2

In this region, we expand around the second saddle S_2 and we obtain

$$\begin{aligned} N \text{Re} S_{T_2} &= N[(a_0^2 + (b_0 + 2)^2 + |a|^2)2\mathcal{E}_i^2 + V_0^2 \\ &\quad + 2[(b_0 + 2)2\mathcal{E}_r\mathcal{E}_i + a_0(\mathcal{E}_i^2 - \mathcal{E}_r^2)]V_0] + O(\|X - S_2\|^3)N \\ &= N(X - S_2)^* H (X - S_2) + O(N^{-\delta}) . \end{aligned} \quad (4.127)$$

The bounds on $N R_2$ and R_3 work as before. For $N^2 R_1$ we have

$$I_1 = 0 + O(\|X - S_2\|) , \quad (4.128)$$

$$\left[\frac{A_0}{(A_0^2 - |a|^2)} + \mathcal{E} \right] = O(\|X - S_2\|) , \quad (4.129)$$

so that again $N^2 R_1 = O(1/N)$.

Bound in the region T_3

We have to bound $f(X) = |e^{-N \text{Re} S(X)} R(X)|$. We shall prove

$$f(X) = |e^{-N \text{Re} S(X)} R(X)| \leq c e^{-N^{1/3-3\delta}} . \quad (4.130)$$

The function $R(X)$ is a rational fraction and to bound it we must take care of the possible zeroes of the denominator. This denominator is $(A_0^2 - |a|^2)^2 (iB_0)^2$.

But $(A_0^2 - |a|^2)$ has no zeroes thanks to the imaginary part \mathcal{E}_i . iB_0 in contrast vanishes on the submanifold $b_0 = -\mathcal{E}_i$ and $V_0 = -\mathcal{E}_r$, but the factor $(iB_0)^{2N}$ that we put in S also vanishes. So we are lead to write

$$e^{-NS(X)} R(X) = e^{-(N-2)S(X)-2S_2(X)+\ln G_a} R'(X) \quad (4.131)$$

where

$$G_a = (-\mathcal{E}^* + a_0 + V_0)^2 - |a|^2 \quad (4.132)$$

$R'(X) = R(X)(iB_0)^2$ is now a rational fraction with a non-vanishing denominator, and

$$S_2(X) = a_0^2 + b_0^2 + 2\mathcal{E}(a_0 - ib_0) + |a|^2 + V_0^2. \quad (4.133)$$

On the compact region K we have

$$e^{-2S_2(X)+\ln G_a} \leq K_1 \quad (4.134)$$

for some constant K_1 , and

$$|R'(x)| \leq cN^{5/2}(1 + \|X\|)^d \quad (4.135)$$

for some constants c and d .

So it is sufficient to prove that in that region

$$(N-2)\text{Re}S_{T_3} \geq c(N-2)\frac{1}{N^{2/3+2\delta}} = c'N^{1/3-2\delta} \quad (4.136)$$

for some small positive constant c and c' . Indeed we have to check where $\text{Re}S$ can reach its minimum. This can only be on a critical point or on the boundary. But the only critical points in T_3 are S_3 and S_4 , where we have the much stronger bound $N\text{Re}S \geq 0.15N$. On the boundary with T_1 and T_2 the previous Hessians approximations are still valid and give the bound (4.136). Finally on the outer boundary the bound is even better (see below).

Bound in the region T_4

When we are far enough from all saddle points we can apply a rough bound to the logarithms in the action

$$e^{N[\frac{1}{2}\ln F_a - (N-2)\ln F_b]} \leq e^{+K'N(\log \|X\|)} \quad (4.137)$$

for K' big. Moreover

$$e^{+K'N(\log \|X\|)} \leq e^{+N\|X\|^2/4} \quad (4.138)$$

taking K (the parameter defining region T_4) large enough, since recall that in T_4 $\|X\| \geq K$.

Finally in T_4 it is easy to check that for large enough K

$$(N-2)(a_0^2 + b_0^2 + 2b_0\mathcal{E}_i - 2a_0\mathcal{E}_r + |a|^2 + V_0^2) \geq (1/2)N\|X\|^2 \quad (4.139)$$

so that remembering (4.102) and the bound (4.135) on R' we can bound

$$\int_{T_4} dX e^{-N\text{Re}S(X)} |R(X)| \leq \int_{\|X\| > K} e^{-NV_{\text{ert}}X\|^2/4} N^{5/2} K_2 (1 + \|X\|)^d \leq e^{-cN}. \quad (4.140)$$

This completes the proof.

4.4 Bounding the correction due to \mathcal{B}

This is explained in a rather detailed way in the last section 6, of [1], which for completeness we roughly reproduce here.

The quartic term \mathcal{B} should be small as $1/N$ compared to the other ones since it contains only one sum over sectors α , not two independent sums over α and β . However it has the wrong sign, so although statistically small, it is dangerous at large fields. The solution is to treat it by some kind of small/large field expansion. A Taylor expansion with integral remainder is written successively for each of the N sectors appearing in the sum for \mathcal{B} :

$$\mathcal{B} = \sum_{\alpha} \mathcal{B}_{\alpha}. \quad (4.141)$$

To first order the expansion gives

$$e^{-\lambda^2 \mathcal{B}_{\alpha}} = 1 - \int_0^1 \lambda^2 \mathcal{B}_{\alpha} e^{-\lambda^2 \mathcal{B}_{\alpha}} e^{t\lambda^2 \mathcal{B}_{\alpha}} dt. \quad (4.142)$$

This Taylor expansion either suppresses \mathcal{B}_{α} from the exponential of the action or generates a remainder term

$$R_{\alpha} = - \int_0^1 \lambda^2 \mathcal{B}_{\alpha} e^{-\lambda^2 \mathcal{B}_{\alpha}} e^{t\lambda^2 \mathcal{B}_{\alpha}} dt. \quad (4.143)$$

The rough line of argument is as follows: either the set P of sectors where \mathcal{B}_{α} is not suppressed is small or it is large. Let $p = |P|$. If p is large, we have gained such a small probabilistic factor that we can treat this case by a rough translation of the V contour, like for large field cubes of section (3.3) (regime $|p^2 - E| \gg \lambda^2$). No saddle point analysis is necessary.

If p is small, we have to perform the saddle point analysis, but we can restrict it to the sum of sectors outside P where it is essentially the same analysis than previously but with only $N' = N - p$ sectors. We have to treat also the coupling between the sectors in and outside P , and this is a delicate point, since the ultimate reason for which these couplings are small is the supersymmetry of the underlying field theory.

Now what "small" or "large" means for p ? It means p smaller or larger than some function $p(N)$. Since we expect a small factor of order $1/N$ for each sector in P (through the lacking sum over β), and a rough imaginary translation on V costs K^N , we need $K^N e^{-p(N) \log N} \ll 1$.

Therefore we choose for $p(N)$ the integer part of $N/\sqrt{\log N}$. The expansion is stopped at order $p(N) = p$. This means that we write:

$$e^{-\lambda^2 \mathcal{B}} = 1 + \sum_{P \subset [1, \dots, N] \text{ } 0 < |P| \leq p(N)} R_P, \quad (4.144)$$

where

$$\begin{aligned}
R_P &= \prod_{\alpha \in P} R_\alpha && \text{if } |P| < p(N), \\
R_P &= \prod_{\alpha \in P} R_\alpha \prod_{\alpha > \max P} e^{-\lambda^2 \mathcal{B}_\alpha} && \text{if } |P| = p(N).
\end{aligned} \tag{4.145}$$

Let Q be the complement of P in $[1, \dots, N]$. The term 1 was treated in the previous sections. The remainders terms R_P must be shown to be $O(1/N)$ as $N \rightarrow \infty$. Let $R_{p(N)} = \sum_{P \subset [1, \dots, N], |P|=p(N)} R_P$ be considered first. For this term we said that it is not necessary to perform any saddle point analysis. It is sufficient to return to the treatment of subsection 3.3, so we undo the V integration: all the $e^{-\lambda^2 \mathcal{B}_\alpha}$ -terms are recombined with the \mathcal{A} -term to reproduce the initial functional integral (4.72). However we cannot exactly reproduce the initial functional integral over V , since some $e^{-\lambda^2 \mathcal{B}_\alpha}$ -factors are missing or appear with reduced weights in (4.145). This means that there remains quartic correction terms $e^{t\lambda^2 \mathcal{B}_\alpha}$ or $e^{\lambda^2 \mathcal{B}_\alpha}$. The important remark is that the bosonic part of these terms has now the right sign ! Therefore they can be represented as a well defined functional integral over a new auxiliary field W_α . For instance

$$\begin{aligned}
\exp(-2\lambda^2 t S_\alpha^+ S_{-\alpha} S_{-\alpha}^+ S_\alpha) &= \exp(-2\lambda^2 t |S_\alpha S_{-\alpha}|^2) \\
&= \int dW_\alpha d\bar{W}_\alpha e^{-|W_\alpha|^2} \exp(i\lambda\sqrt{2t} W_\alpha S_\alpha \bar{S}_{-\alpha} + cc).
\end{aligned}$$

With slightly condensed notations, this leads to

$$\begin{aligned}
R_{p(N)} &= \sum_{P \subset [1, \dots, N], |P|=p(N)} \int S^+ S \left(\prod_{\alpha \in P} \int_0^1 \lambda^2 \mathcal{B}_\alpha dt \right) \\
&= \times \exp(i\Psi^+(E + \lambda V + \lambda\sqrt{2t}W)\Psi) d\Psi^+ d\Psi d\mu(V, W). \tag{4.146}
\end{aligned}$$

Then a complex translation $V_0 \mapsto V_0 \pm i\lambda^{-1}$ is performed, with the same sign as the imaginary part of E in order to avoid crossing of singularities. In other words

$$\int_{-\infty}^{+\infty} e^{-V_0^2} F(V_0) dV_0 = \int_{-\infty}^{+\infty} e^{-V_0^2} e^{-2i\lambda^{-1} V_0^2} e^{\lambda^{-2}} F(V_0 + i\lambda^{-1}) dV_0. \tag{4.147}$$

The functional integral can now be bounded by its absolute values everywhere, namely the following contributions are bounded

- by 2^N , for the sum over P , that is the total number of subsets of $[1, N]$;
- by 1, for the integrals such as $\int_0^1 dt$;
- by 1, for the oscillating imaginary integrals;

- by Gram's inequality for fermions or the Schwarz inequality for Bosons, for the remainders terms \mathcal{B}_α ;
- by 1, for every propagator since the imaginary translation in V_0 has created an imaginary part proportional to the identity in the denominator of the Green's function.

This means that each \mathcal{B}_α term gives rise as expected to a small factor $\lambda^2 = 1/2N$ for each sector α in P , hence altogether we have a small factor $1/N^{p(N)}$. The two source terms are bounded by 1. The normalization determinants are then easily bounded by c^N , even without using the supersymmetry cancellations, since the operators considered are bounded in a finite $2N$ dimensional space thanks to the imaginary part of E which is no longer infinitesimal. Combining all factors leads to

$$R_{p(N)} \leq c^N e^{-cN\sqrt{\log N}}, \quad (4.148)$$

showing that this correction term is indeed small.

It remains to treat the terms with $1 \leq |P| \leq p(N)$. To bound these terms a mean-field analysis will be performed like in subsection (3.2), introducing the a and b fields, but we said it should apply only to the sectors of the theory in the complement Q of P . The functional integral to be bounded for a single term is (4.145). Now, the $e^{-\lambda^2 \mathcal{B}_\alpha}$ -terms are recombined only for $\alpha \in P$ with the \mathcal{A} term to reproduce the initial functional integrals over the V fields and the terms $e^{t\lambda^2 \mathcal{B}_\alpha}$, also for $\alpha \in P$, are again given by defined integrals over new auxiliary fields W_α . Finally, the quartic terms, with sector sums reduced to Q , are treated exactly as in the previous section, hence mean-fields a, a_0, b_0 are correspondingly introduced. This leads to a representation

$$R_P = \int dV_{P,P} dV_{P,Q} d^2a dV_0 da_0 db_0 \prod_{\alpha \in P} R_\alpha e^{\mathcal{L}_Q}, \quad (4.149)$$

where Q is the complement of P in $[1, \dots, N]$. In addition, $V_{P,P}$ is the part of the matrix V corresponding to rows and columns in P , including the new fields of the W type. $V_{P,Q}$ correspond to one entry in P and the other in Q , and the mean field computation is now restricted to Q . The integral over superfields gives rise again to a superdeterminant and an additional correction, whose bosonic part (leaving the Fermionic part to the reader) is of the type

$$\exp [Tr \log(1 + CV)], \quad (4.150)$$

where C is the bosonic part of the R^{-1} matrix for the $N - p$ analogous problem, as in the previous subsection (see eq. (4.78)), and $V = V_{PP} + V_{P,Q}$ is the perturbation. This correction term is bounded by

$$|\exp [Tr \log(1 + K)]| \leq \exp (Tr K + K^* + K K^*). \quad (4.151)$$

Evaluating $C = R^{-1}$ at the saddle point costs a factor $\exp(Np/2N) = e^{2p}$ at most. Each term \mathcal{B}_α naively gives a factor $\lambda^2 = 1/(2N)$ when evaluated,

but this simply compensates the sum over α when p is nonzero but small, so we have to gain an additional $1/N$ factor. Adding a few expansion steps gives such a small additional factor $1/N$ (already for the first non zero value $p = 1$). This can be seen by integrating by parts the superfields in the *vertex* \mathcal{B}_α which has been taken down the exponential. By supersymmetry, the *vacuum graph* corresponding to a contraction of the four fields at the vertex vanishes. This is absolutely necessary since this graph by simple scaling is proportional $1/N$ and cannot have any additional $1/N$ factor. Its vanishing can be checked by hand:

- the self-contractions of the bosonic piece $2S_\alpha^+ S_{-\alpha} S_{-\alpha}^+ S_\alpha$ give a factor $+2$;
- the self-contractions of the boson-fermion piece

$$\left(\sum_{\sigma} S_{\sigma\alpha}^+ \chi_{-\sigma\alpha}^+ \right) \left(\sum_{\sigma'} S_{\sigma'\alpha} \chi_{-\sigma'\alpha} \right),$$

give a factor -2 (since there is one Fermionic loop giving the minus sign and one sum over σ giving a factor 2 only after the contractions);

- the selfcontractions of the term $(1/2) \sum_{\sigma} (\Psi_{\sigma\alpha}^+ \Psi_{\sigma\alpha})^2$ are clearly supersymmetric and also add up to 0.

Consequently, at least one field of the *vertex* \mathcal{B}_α has to contract to the exponential. Performing two contractions in turn, gives always at least a factor p/N^2 at the end instead of the naive $1/N$ factor. Indeed the worst case corresponds to the non trivial contraction term being of the type V_{PQ} . This generates a new factor $1/N$ but a new sum over $\beta \in Q$, which costs $N - p \simeq N$ so nothing is gained yet. But contracting this β field again either generates a diagonal term, hence a new $1/N$ factor, with no new sum, or returns to a $V_{Q,P}$ term. This last situation generates a new $1/N$ factor and a new sum over sectors γ but this time this new sum is restricted to P , so it costs only a factor p instead of N ! Hence at worst, after these two contraction steps, a total factor p/N^2 instead of the naive factor $1/N$ is associated to each vertex, as announced. Now the sum over P costs a total factor $N!/p!(N-p)!$, hence is bounded by $c^p [N/p]^p$. Combining all factors, the sum of contributions of such terms with $p \neq 0$ is bounded by $\sum_{p=1}^{+\infty} [c/N]^p \leq c'/N$. It is therefore at least as small as $1/N$.

This completes our sketch of the content of [1]. The last section in this review is devoted to generalizations that better mimic the symmetries of the 2-dimensional matrix model near degeneracy of the rhombus, using anisotropic sectors.

5 Improved Flip Matrix Models

As we already said in Section 3.2, the isotropic sector model is not optimal for momentum conservation analysis.

Actually we know that the potential $V(\gamma)$, has to be associated to pairs of vectors p, q with $|p| = 1 + O(M^{-j})$, $|q| = 1 + O(M^{-j})$ (using the same notation

as in previous sections) and $p - q = \gamma$. By simple geometric arguments we can see that the angle σ between p and q is then $\sin(\sigma/2) \simeq |\gamma|/2$ and p can move of an angle $\delta\sigma = O(M^{-j}/|\gamma|)$.

- If we have isotropic sectors (of width M^{-j}), then for each $|\gamma|$ there is a set $S(\gamma)$ of $O(1/|\gamma|)$ pairs of sectors α, β (with angle σ between them) associated to the random variable $V(\gamma)$. So when $|\gamma| < 1$ the matrix elements $V_{\alpha\beta}$ for $(\alpha, \beta) \in S(\gamma)$ are no longer independent random variables. In the limit case, $|\gamma| = O(M^{-j})$ all the $O(M^j)$ sectors are associated to the same $V(\gamma)$. This is the diagonal term V_0 that appears in the Isotropic flip model.
- If we still want to work with matrix elements which are all independent random variables we have to introduce anisotropic sectors of width $M^{-j/2}$. In this case the matrix elements are all independent for $\gamma \geq M^{-j/2}$. The case $\gamma < M^{-j/2}$ corresponds to the diagonal terms $V_{\alpha\alpha}$. Here we do expect to have no longer independence, but the problem arises only on the diagonal terms, just as in the Isotropic flip model. The drawback is that now, for each sector α we have also to introduce a dual space decomposition R_α of the spatial cube Δ of side M^j in rectangles of side M^j in the direction parallel to α and $M^{j/2}$ in the orthogonal direction. The random matrix is then indexed by pairs (α, t) where α is a sector and t a dual rectangle.

5.1 The Anisotropic (2d) Flip Matrix Model

As in the previous sections we consider the Anderson model in a spatial cube Δ of side M^j , and we introduce a cut-off in momentum space that selects only an annulus of width M^{-j} around the Fermi surface. We fix then $M^j = \lambda^{-2} = 2N$, $E_0 = 1$ and $E = O(\lambda^2) = O(N^{-1})$.

We then cut the annulus in anisotropic sectors of width $M^{-j/2}$, so that a sector now has width M^{-j} in the radial direction and $M^{-j/2}$ in the tangential direction. Therefore we have $M^{j/2}$ anisotropic sectors instead than M^j isotropic ones. For each sector α then we introduce a partition R_α of the cube Δ in rectangles of side M^j in the direction parallel to α and $M^{j/2}$ in the orthogonal direction.

In this way we obtain a finite dimensional space of dimension $2N = M^j$ with a new index for its orthogonal basis, namely the pairs (α, r, σ) where α runs over the first \sqrt{N} anisotropic directions, $r \in R_\alpha$ runs over the \sqrt{N} rectangles elements of R_α and as previously σ takes two values ± 1 corresponding to the two opposite sectors on the circle. The dimension of the Hilbert space for the matrix model we want to construct is then the same as in the isotropic case.

We list now the columns and rows of our random matrix as before, with sectors from 1 to \sqrt{N} , then from $-\sqrt{N}$ to -1, and for each sector α we list the new rectangular variables r in direct order according to an arbitrary ordering of R_α . Note that the rectangles for α and $-\alpha$ are then the same modulo a

rotation of π . The effect of the rotation is that the ordering of the rectangle is inverted (see Fig.9). This is actually good as it ensures the symmetry of the matrix around the anti-diagonal.

Finally, as in the case of the isotropic sectors, we discretize the potential and build a random matrix model, that will be called Anisotropic Flip Matrix Model (AFMM).

5.1.1 Construction of the matrix model

To construct the matrix we introduce a discretization of the phase space for V , in such a way to ensure that:

- the off-diagonal matrix elements are independent random variables with covariance one.
- the dependence among the diagonal terms is explicitly extracted. This means each $V_{\alpha\alpha}$ is a sum of independent random variables $V_{\alpha\alpha} = V_0 + V_1 + \dots + V_{j/2}$ where V_0 appears in all diagonal terms and $V_{j/2}$ only in one.

Note that as V is restricted to act on the shell $|p^2 - 1| = O(M^{-j})$, the momentum γ for V is restricted to $|\gamma| \leq 2 + O(M^{-j})$. In position space V is restricted to the cube Δ .

To construct the off-diagonal matrix elements we divide the momentum region γ in three parts:

- a *forward-momentum* region corresponding to $M^{-j/2} < |\gamma| < 1$ (sector pairs at an angle $M^{-j/2} < \sigma < \pi/3$),
- a *backward momentum* region corresponding to $1 \leq |\gamma| \leq 2 + O(M^{-j})$ (sector pairs at an angle $\pi/3 \leq \sigma \leq \pi + O(M^{-j})$),
- a *0-momentum* region corresponding to $|p| \leq M^{-j/2}$ ($\sigma \leq M^{-j/2}$); this last region gives the diagonal terms of the matrix

In these regions the discretization of the phase space for V must be different. To see that let us consider a pair of sectors α, β with two vectors inside $p \in \alpha, q \in \beta$ and $\gamma = \alpha - \beta$. As p and q move inside their sectors the variation for γ is given by

$$\delta\gamma_{\parallel} = M^{-j/2} \cos\left(\frac{\sigma}{2} + M^{-j/2}\right) \quad \delta\gamma_{\perp} = M^{-j/2} \sin\left(\frac{\sigma}{2} + M^{-j/2}\right). \quad (5.152)$$

as $M^{-j/2}$ is the sector size. Now we distinguish four limit situations (see Fig.5-6)

1. When $|\gamma| \simeq 1$ ($\sigma \simeq \pi/3$) then $\delta\gamma_{\parallel} \simeq \delta\gamma_{\perp} = O(M^{-j/2})$ so we should cut momentum space in cubes of side $M^{-j/2}$. This corresponds to γ_2 in Fig.6.

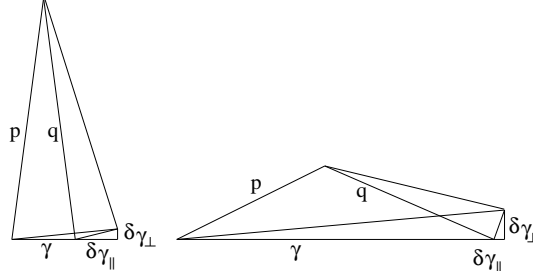


Figure 5: Variation of γ as p and q move inside their sectors

2. When $|\gamma| \simeq M^{-j/2}$ ($\sigma = O(M^{-j/2})$) we are almost on the diagonal part of the matrix (γ_1 in Fig.6). The value of γ is fixed with a precision much higher than $M^{-j/2}$ in the perpendicular direction, that is $\delta\gamma_{\parallel} = O(M^{-j/2})$ while $\delta\gamma_{\perp} = O(M^{-j})$. In this region we must then use rectangles of width $M^{-j/2}$ in the radial direction (that of γ_{\parallel}) and M^{-j} in the orthogonal direction.

3. When $|\gamma| = 2 - M^{-j}$ ($\sigma \simeq \pi$) the situation is reversed. The vector γ is fixed with high precision in the parallel direction. For $\delta\gamma_{\parallel}$ we have

$$\delta\gamma_{\parallel} = M^{-j/2} \sqrt{1 - (|\gamma|/2)^2} = M^{-j/2} \sqrt{M^{-j} - M^{-2j}/4} = O(M^{-j}) \quad (5.153)$$

while $\delta\gamma_{\perp} = O(M^{-j/2})$. So this time we must use rectangles of width M^{-j} in the radial direction (that of γ_{\parallel}) and $M^{-j/2}$ in the orthogonal direction (see γ_3 in Fig.6).

4. Finally when $|\gamma| < M^{-j/2}$ ($\sigma < M^{-j/2}$) we are on the diagonal terms of the matrix. Here too we have two extreme cases. For $|\gamma| \simeq M^{-j/2}$ the momentum space should be partitioned as above, and there is still only one sector pair (so only one matrix element $V_{\alpha\alpha}$ associated to the corresponding random variable. But when $|\gamma| \leq M^{-j}$, then we have only one cube of side M^{-j} around the value $\gamma = 0$ and the corresponding random variable appears in every diagonal term.

In order to interpolate between these extreme situations we have to introduce shells where $|\gamma|$ is approximately constant. In each shell then we introduce cubes or rectangles depending on the value of $|\gamma|$. The details are shown below.

Forward-momentum. This region corresponds to the annulus between $|\gamma_1|$ and $|\gamma_2|$ in Fig.6 and is divided into shells according to an index k , $k = 0, \dots, j/2 - 1$ so that $M^{-k-1} \leq |p| \leq M^{-k}$.

Each forward-shell is now further divided into rectangular momentum cells of length $O(M^{-j/2})$ in the direction parallel to the center of the cell, and of width $M^{-k-j/2}$ in the direction perpendicular to the center of the cell.

Finally, for each such cell, the cube Δ is divided into corresponding dual tubes τ of length $M^{k+j/2}$ in the direction perpendicular to the center of the cell, and of width $M^{j/2}$ in the direction parallel to the center of the cell. Our normalization is that each random variable for a pair (γ, τ) , which corresponds to a volume 1 in phase space has covariance 1. These variables are independent Gaussian variables.

Backward momentum region. The backward region (region between $|\gamma_2|$ and $|\gamma_3|$ in Fig.6) is also divided into shells labeled by a similar index $k = 0, \dots, j/2$, but the backward k -shell corresponds now to momenta $2 - M^{-2k} \leq |p| \leq 2 - M^{-2k-2}$ for $k < j/2$. The last shell is simply $2 - M^{-j} \leq |p| \leq 2 + M^{-j}$. Each shell is now further partitioned into $O(M^{j-k})$ cells γ of width $M^{-j/2}$ in the direction perpendicular to the center of the cell, and length $M^{-j/2-k}$ in the direction parallel to the center of the cell. The cube Δ is again divided into dual tubes τ of width $M^{j/2}$ in the direction perpendicular to the center of γ and length $M^{j/2+k}$ in the parallel direction. Each corresponding random variable $V_{\gamma,\tau}$ corresponds again to a different discretized independent random variable for the potential V with covariance 1 since it corresponds to a volume 1 in phase space.

0-momentum region. The last shell is simply $|p| \leq M^{-j/2}$. The treatment of this region is subtle because the anisotropic sectors are not a basis very well adapted to this zero momentum region. We remember that the coefficients cannot be totally independent because in the isotropic basis (treated previously) there is a diagonal term proportional to a single variable $V_0 Id$ where Id is the identity matrix, and this term must be present in any basis. They are not totally dependent either, so the best description consist in further splitting the phase-space corresponding to the 0-momentum region into unit volumes in phase-space.

Therefore we split the 0-momentum region into shells $Sh_{0,k'}$, in which the momentum size is between $M^{-j+k'-1}$ and $M^{-j+k'}$ (for $k' = 0$ the lower bound being simply 0). Each such shell is divided into $O(M^{k'})$ cells γ' of length $O(M^{-j+k'})$ in the direction parallel to the center of the cell, and M^{-j} in the perpendicular direction. The central shell $k' = 0$ corresponds to a single cell (the "true" zero momentum region).

The size of the cells ensures that, given a shell k' and a matrix element $V_{\alpha\alpha}$ only one cell in the shell contributes to it. On the other hand the same cell may contribute to different diagonal terms. In the limit $k' = 0$ the same cell appear in all diagonal terms.

For each cell we divide the cube into dual tubes τ' , with length M^j in the direction perpendicular to the center of the cell, and width $M^{j-k'}$ in the parallel direction. Remark that the total number of pairs (γ', τ') which pave the phase space of this 0-momentum region remains $O(M^j)$ as it should be, and that to each such pair is associated a random variable of covariance 1.

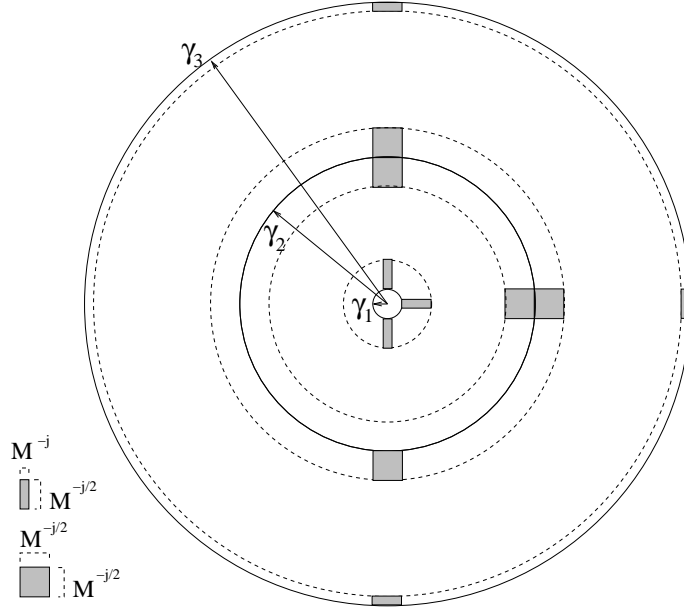


Figure 6: Different decomposition depending on $|\gamma|$: for $|\gamma| = |\gamma_3| \simeq 2$ we have to use rectangles of side M^{-j} in one direction and $M^{-j/2}$ in the other direction, for $|\gamma| = |\gamma_2| \simeq 1$ we have to use cubes of side $M^{-j/2}$ and for $|\gamma| = |\gamma_1| \simeq M^{-j/2}$ we have to use again rectangles, but exchanging the sides

The resulting random matrix. Now to model the rhombus rule we have to understand the random variable V coupling two anisotropic pairs (α, t) and (β, t') . A momentum cell γ of V defines uniquely four possible pairs α, β (up to nearest neighbors, neglected). Indeed we still have Hermiticity and spin flip:

$$V(\alpha, \beta) = \bar{V}(\alpha, \beta) = V(-\beta, -\alpha) = \bar{V}(-\alpha, -\beta) . \quad (5.154)$$

Accordingly there are about $O(M^k)$ pairs α, β which have their sum in the forward or backward shell of index k . We consider then two tubes t, t' and remark that their intersection is either empty or roughly of width $M^{j/2}$ and length $M^{j/2+k}$. Furthermore as soon as k increases, the direction of α and β begins to collapse. This applies both to forward and backward scattering.

When they intersect, their intersection is a tube of length $M^{k+j/2}$ and width $M^{j/2}$ and there is therefore a single independent variable $V_{\gamma, \tau}$ of covariance 1 both in the forward scattering case and in the backward scattering case corresponding to this intersection (see Figure7).

When the two tubes miss (see Figure 8), we put a zero in the corresponding random matrix.

This describes the random matrix if we complete by Hermiticity and symmetry around antidiagonal (spin flip), except for the diagonal blocks $(\alpha, t; \alpha, t')$ which correspond to the 0-momentum region for V . The coefficients of these

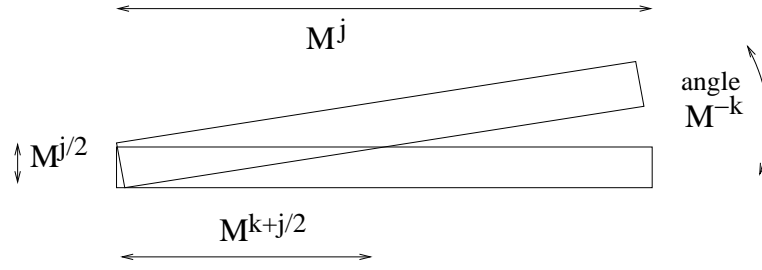


Figure 7: Intersecting tubes

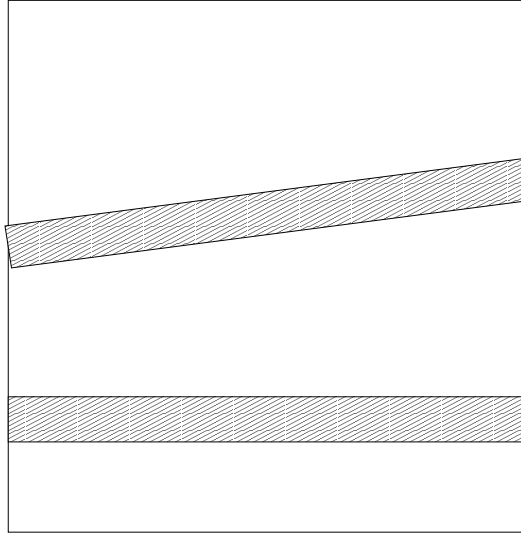


Figure 8: Two missing tubes in a box

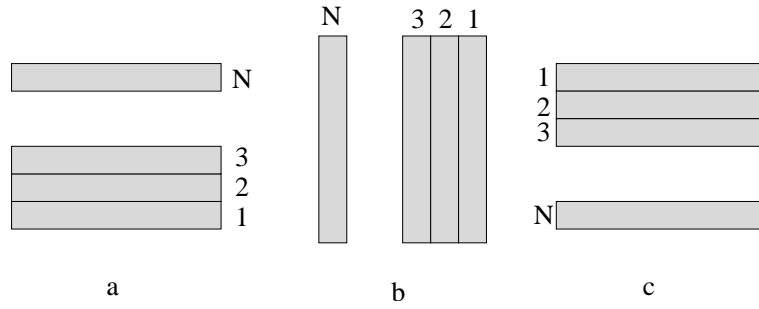


Figure 9: Ordering of the spatial tubes after a rotation of $\pi/2$ (b) or π (c)

blocks are 0 except on the pure diagonal $t = t'$, since two distinct parallel tubes t and t' are disjoint. (Similarly the antidiagonal blocks corresponding to extreme backward scattering $\alpha = -\beta$ are purely antidiagonal (0 except on the anti-diagonal). But the antidiagonal is made of independent random variables of covariance 1, while the diagonal is more complicated). As we have seen above, a diagonal term is represented as a sum of independent random variables each corresponding to a different shell k' .

Let (α, t) be a given anisotropic sector and tube, and k' a shell index. We call $\gamma'(\alpha, k')$ the cell of the shell of index k' perpendicular to the center of α , and we call $\tau'(t, k')$ the corresponding associated tube containing t . The pair $(\gamma'(\alpha, k'), \tau'(t, k'))$ corresponds to a single random variable of scale k' associated to (α, t) .

The diagonal coefficient of our random matrix is then simply

$$M_{(\alpha, t); (\alpha t)} = \sum_{k'=0}^{j/2} V_{\gamma'(\alpha, k'), \tau'(t, k')} . \quad (5.155)$$

Remark that conversely for a given (γ', τ') in shell k' there are roughly $M^{j/2-k'}$ tubes t in τ' , and roughly $M^{j/2-k'}$ cells α giving the same γ' , hence $M^{j-2k'}$ pairs (α, t) giving the same (γ', τ') . This means that a given random variable $V_{\gamma', \tau'}$ appears at $M^{j-2k'}$ different diagonal places in the matrix. For $k' = 0$ the single random variable, also called V_0 appears everywhere. At the other end, for $k' = j/2$ the variables $V_{\gamma', \tau'}$ appear essentially at a single place.

5.2 The hierarchical approximation

This approximation consists in somewhat simplifying the intersection rule for t and t' and the angular condition on α and β . For $(\alpha \neq \beta)$ instead of considering that the pair is of class k if their sum lies in the k -th shell, we divide the set of anisotropic sectors into M^k packets for each value of k , $k = 0, \dots, j/2$ (each packet containing $M^{j/2-k}$ sectors) and we replace the rule on the sum γ to be in shell k by the simpler "hierarchical" condition that α and β belong to the same packet of scale k but not to the same of scale $k + 1$.

In the same way we simplify the intersection rule for the tubes. We divide for each sector α the set of tubes T_α into M^k packets, according to translation in the direction perpendicular to α and label these packets with an index $r_{\alpha l}^{(k)} = 1, \dots, M^k$. Each packet corresponds itself to a larger tube of width M^{j-k} , hence contains $M^{j/2-k}$ anisotropic tubes of width $M^{j/2}$. Let us consider now two tubes t, t' for a pair $(\sigma_\alpha \alpha, \sigma_\beta \beta)$ of category k , where $\sigma_\alpha = \pm 1$ says if α is in the upper half or lower half of the circle. Then we slightly simplify the intersection rule by saying that, if $\sigma_\alpha = \sigma_\beta$, then t, t' intersect if their packet index of degree k is the same: $r_\alpha^{(k)} = r_\beta^{(k)}$ and miss completely otherwise. On the other hand, when $\sigma_\alpha = -\sigma_\beta$, for instance $\sigma_\alpha = 1, \sigma_\beta = -1$, then the packets in R_β have performed a rotation by an angle π , and the ordering is reversed (see Fig.9). In this case we say that t, t' intersect if their packet index satisfy $r_\alpha^{(k)} = M^k - r_\beta^{(k)}$.

Finally we take $M = 2$, which also simplifies the computation. In the limit $N \rightarrow \infty$ the density of states of the real problem should be similar to the limit of this hierarchical limit.

The matrix can then be represented as a set of $\sqrt{N} \times \sqrt{N}$ blocks. Each block corresponds to a fixed sector pair, and the variables in the block give the couplings between the tubes. In the blocks corresponding to $k = 1$ (both in the backward and forward region) all sectors intersect, so all elements are independent random variables. For $k > 1$ the block has a block diagonal structure, with $2^{j/2-k} \times 2^{j/2-k}$ blocks along the diagonal or the antidiagonal, depending if we are considering the forward or the backward region. The elements inside the blocks are independent random variables, the elements outside are zero. In the limit when $k = j/2$ the blocks have non zero elements only on the diagonal or antidiagonal. In the first case, the elements are no longer independent variables.

Note that, from the definition of forward and backward shells, the two $k = 0$ shells cover angles $\pi/6 < \sigma < 2/3\pi$ approximately. In this region $k = 0$ then all tube packets intersect and the corresponding block of the matrix has no zeros. Therefore, for each line in the matrix, at least half of the blocks have no zeros. For $k > 0$ (both in the backward and forward region) we are considering only small angles. As a result the matrix looks like Fig.10.

We still have to say what are the diagonal terms $V_{\alpha,t;\alpha,t}$. We may consider only the first \sqrt{N} sectors as the others are obtained by flip symmetry. The diagonal \mathcal{D} can then be written as a sum of diagonal matrices

$$\mathcal{D} = \sum_{k'=0}^{j/2} \mathcal{D}_{k'} . \quad (5.156)$$

For $k' \neq k''$ the corresponding random variables are independent. For a fixed k' $\mathcal{D}_{k'}$ is again a diagonal matrix, made of $2^{k'}$ diagonal blocks $B(k')_r$ $r = 1, \dots, 2^{k'}$. Each block $B(k')_r$ actually selects a subset of $2^{j/2-k'}$ sectors that are associated to the same set of random variables. For $k' = 0$ there is only one block as $\mathcal{D}_0 = V_0 Id$. In the opposite case $k' = j/2$ there are $2^{k'}$ as all diagonal elements are independent random variables. As an example, for $k' = 1$ we have only two blocks

$$\mathcal{D}_1 = \begin{pmatrix} B(1)_1 & 0 \\ 0 & B(1)_2 \end{pmatrix} . \quad (5.157)$$

Then a single block $B(k')_r$ is built from $2^{j/2-k'}$ identical sub-blocks $V(k', r)$, one for each sector. This sub-block $V(k', r)$ actually identifies the set of independent random variables associated to this set of sectors. For $k' = 0$ the single block $B(0)$ has $2^{j/2}$ sub-blocks, that is all sectors are associated to the same set of variables.

Finally, we have to write the $\sqrt{N} \times \sqrt{N}$ block $V(k', r)$ which corresponds to the tube indices. This block contains $2^{k'}$ independent random variables. In order to distribute them on the diagonal we introduce a partition of the set $\{1, \dots, \sqrt{N}\}$ in $2^{k'}$ sets of $2^{j/2-k'}$ indices, defined by $I = \cup_{l=1}^{2^{k'}} I_l$ and

$$I_l = \{i \mid 2^{j/2-k'}(l-1) < i \leq 2^{j/2-k'}l\} . \quad (5.158)$$

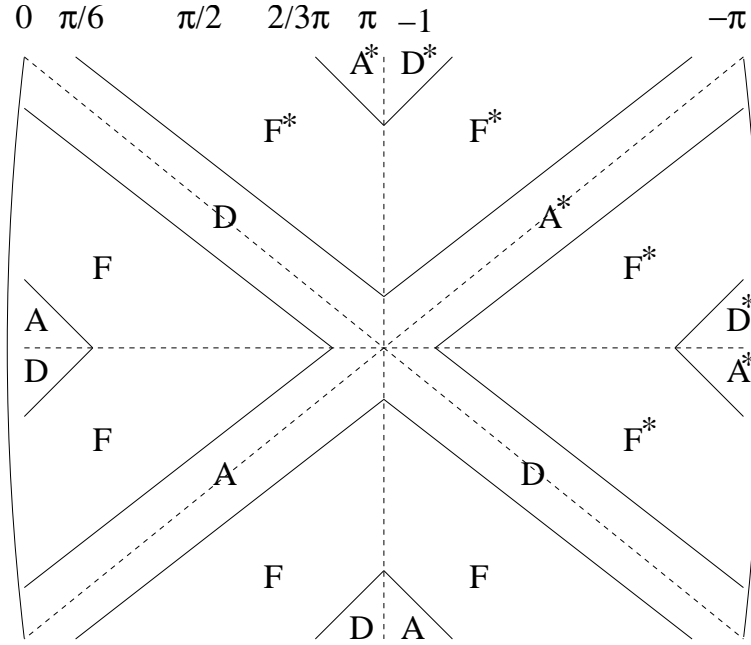


Figure 10: Form of the random matrix after the hierarchical approximation; the F represent the areas where no zeros appear, D and A are the areas where only blocks on the diagonal or antidiagonal are non zero. The size of the block is very big as we are near to the border with the F region and reduce to one as we go towards the diagonal and antidiagonal

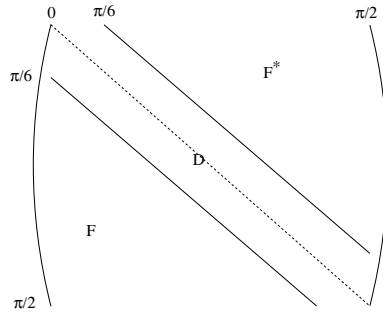


Figure 11: Form of the random matrix if we select only the first $\sqrt{N}/2$ sectors

Now to each $l = 1, \dots, 2^{k'}$ we associate an independent random variable $V(k', r)_l$. With this notations

$$V_{ii} = V(k', r_{k'})_{l'_k} \quad \text{if } i \in I_l. \quad (5.159)$$

Again we see that in the case $k' = 0$ we have only one set I so that there is only one independent random variable.

5.2.1 Toy model

To understand how the analysis is going to change with respect to the isotropic model we first consider a simplified model, where we restrict to the sectors $1, \dots, \sqrt{N}/2$, that is to angles $0 \leq \sigma \leq \pi/2$. The matrix then looks like Fig11.

We can construct now this matrix in a hierarchical way.

The first hierarchical step is $k = 1$, with two packets of sectors, and two packets of tubes. If we list the indices of the first packet first and second packet second, and then within each sector packet we list the tube indices with first packet of tubes first, this cuts the matrix into 16 blocks. Then we put random independent coefficient (with the Hermiticity condition) everywhere except in the two upper blocks B_{12} and B_{34} which we fill with zeros. (We complete by Hermiticity so that lower blocks B_{21} and B_{43} are also zero). These zeros correspond to the first group of missing tubes. In this way we get an $N \times N$ called matrix $M_1(N)$

$$M_1(N) = \begin{pmatrix} B_{11} & 0 & B_{13} & B_{14} \\ 0 & B_{22} & B_{23} & B_{24} \\ B_{13}^* & B_{23}^* & B_{33} & 0 \\ B_{14}^* & B_{24}^* & 0 & B_{44} \end{pmatrix} = \left(\begin{array}{cc|cc} B & 0 & & \\ 0 & B & & \\ \hline & & B^* & \\ B & & B & 0 \\ \hline & & 0 & B \end{array} \right) \quad (5.160)$$

where $B_{ii}^* = B_{ii}$ for all $i = 1, \dots, 4$ and each block B_{ij} is a $N/4 \times N/4$ matrix.

Then for $k = 2$, the 4 previous diagonal blocks of $M_1(N)$ are themselves replaced by $M_1(N/4)$. This defines $M_2(N)$, which is a sixteen by sixteen matrix.

$$\left(\begin{array}{cc|cc|cc|cc} B & 0 & & & & & & \\ 0 & B & & & & & & \\ \hline & & B^* & & & & & \\ B & & B & 0 & & & & \\ \hline & & 0 & B & & & & \\ 0 & & & & B & 0 & & \\ & & & & 0 & B & & \\ \hline & & & & & & B^* & \\ & & & & & & B & 0 \\ & & & & & & 0 & B \\ \hline & & & & & & & \\ B & & & & & & & \\ \hline & & & & B & 0 & & \\ & & & & 0 & B & & \\ \hline & & & & & & B^* & \\ & & & & & & B & 0 \\ & & & & & & 0 & B \end{array} \right)$$

We iterate. To get $M_k(N)$ we replace each of the 4^{k-1} diagonal blocks of $M_{k-1}(N)$ by $M_1(N/4^{k-1})$. We stop at $k = j/2$. Then the diagonals blocks of size 1 have been reached.

Finally in the last step we change the diagonal coefficients of $M_{j/2}(N)$. We replace each of them by the appropriate sum of independent variables according to the analysis of the zero momentum region. Again this is done in a hierarchical way. At $k = 0$ we put a V_0 at every diagonal point. At $k = 1$ we split the diagonal in two parts. To all terms V_{ii} with $i \leq N/2$ we add the variable V_1^1 , to all others V_1^2 . To iterate that we assign an ordering $r_k(i)$. For $k = 1$ we have $r_k(i) = 1$ for all $i \leq N/2$ and $r_k(i) = 2$ otherwise. At $k = 2$ we split in four sets and iterate.

The diagonal coefficient $V_{i,i}$ of the final matrix is therefore

$$V_{ii} = V_0 + \dots + V_k^{r_k(i)} + V_{j/2}^{r_{j/2}(i)} \quad (5.161)$$

where at each scale k there is 4^{k-1} variables V_k^r , each serving for $N/4^{k-1}$ coefficients, and $r_k(i) = 1, \dots, 4^{k-1}$ is simply the label for the variable of the cell in the shell k associated to i .

The rest of the matrix remains unmodified, with independent variables.

So the matrix is made of

- independent random variables at each non-diagonal place, except where the zeros of $M_{j/2}(N)$, lie, which correspond to tubes missing each other,
- a more complicated sum of random variables at each diagonal place. Remark in particular that this sum always contains the V_0 variables, common to all the diagonal.

Again the density of states for this matrix should follow some semi-ellipse Wigner's law when $N \rightarrow \infty$. We have to reproduce the computation of the previous section but with additional fields which complicate the picture. Each set of zeros should indeed be represented by a new pair of complex fields which suppress the off diagonal blocks with zeros in the quartic form after integrations of V .

When searching for a main saddle point, an iterative solution seems doable by plugging the smaller blocks solutions into large means fields, until the last one for which we should get $\nu(E) = C_k \sqrt{E_k^2 - E^2}$ where E_k lies between $\sqrt{2}$ and $\sqrt{3}$ [31].

5.3 A Hierarchical Isotropic matrix with correct Momentum conservation rule

An alternative is to stick to the initial basis of isotropic sectors but to take seriously into account the necessary modification of the rhombus rules for degenerate rhombuses.

The approximation in which each random variable corresponds roughly to a single pair of momenta, up to Hermiticity and flip is indeed not correct. When

the momentum of V is of size M^{-k} (forward) or $2 - M^{-2k}$ (backward), there are roughly M^k pairs that have this transfer momentum.

Taking again $M = 2$ the hierarchical approximation to this correct conservation rule consists in identifying blocks of 2^k by 2^k coefficients to a single random variable as approaching the diagonal (or anti-diagonal in the version with the flip symmetry).

In this point of view the hierarchy is indexed by k from 0 to j (not $j/2$). This model should again give the same limit as the previous one when $j \rightarrow \infty$.

6 The isotropic 3D Model

We would like to conclude with a remark concerning the three dimensional model. The $U(1)$ symmetry which generalizes the flip symmetry in that case has orbits which are quite complicated to describe because there is no simple labeling of sectors on the sphere which respect the rotation invariance of the sphere. For instance labeling according to spherical coordinates introduces a preferred polar axis. So here again it may be instructive to mimic this $U(1)$ symmetry by treating first some hierarchical approximations in which the $U(1)$ symmetry is replaced by a discrete Z_p symmetry, with $p = 2^k$. For instance such symmetries could act in a simple way on coefficients, by folding k times the upper triangle around its bisecting line, and identifying the 2^k coefficients at the same position in all the layers. Then letting $p \sim \sqrt{N}$ and $N \rightarrow \infty$, it should be possible to prove again through a supersymmetric analysis that the density of states of such a model tends to Wigner's law.

Such a result would certainly increase our confidence that the density of states for the true three dimensional Anderson model falls into the same universality class than the GUE and obeys Wigner's semi-circle law.

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